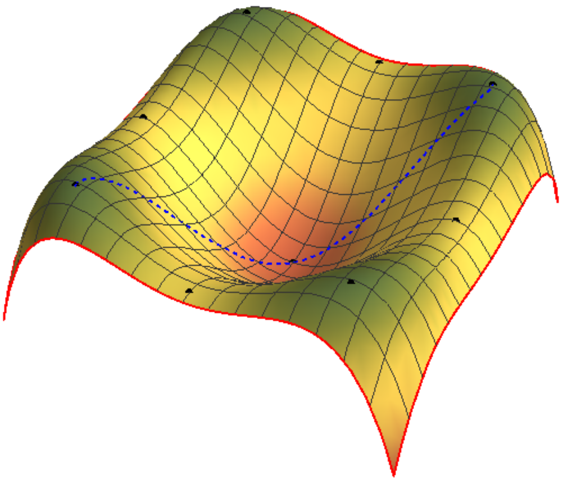
Yang-Mills Theory on Coset Spaces with antisymmetric Torsion

Master Thesis



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1 Introduction

The first field theory encountered by physicists is probably Maxwells electrodynamics. It is a gauge theory, which means that the Lagrangian of the theory is invariant under certain group transformations, in this case the abelian group U(1). This theory already has a wide area of application, especially in its quantized form. However, it is not sufficient to describe a complete model of particle physics.

It turned out that in order to accurately describe the experimental results one needs a theory that has a more complicated gauge group, one that is non-abelian. Such a theory was first discussed by Yang and Mills in 1954 [1] with the gauge group SO(3), although their ansatz failed to give results that were consistent with experimentation; it predicted only massless particles. A few years later, Glashow formulated a theory combining electromagnetism and the weak nuclear force into a $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theory [2], that was brought into its modern form by Weinberg and Salam in 1967 [3]. The problem of masslessness was overcome by employing the Higgs mechanism, where the gauge group is spontaneously broken to U(1) by introduction of an additional scalar field. In the mid-1970s the standard model was formulated, giving a unified field theory of electromagnetism and the weak and strong nuclear forces, with the gauge group $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{SU}(3)$. This theory has had remarkable success, accurately describing the experiments carried out in particle accelerators up to today.

There is a fourth fundamental force, gravity. It is best described by the general theory of relativity, first formulated by Einstein in 1915 [4]. The field theories mentioned above give equations for the behaviour of fields that live on some background spacetime. In contrast, general relativity is a theory of spacetime itself. This leads to complications if one tries to incorporate gravity into the standard model. One feature of quantum field theories is that they may contain integrals which evaluate to infinity, which is dealt with by renormalizing the theory, giving these integrals finite values; the problem is that this procedure fails if one tries to apply it to a quantum theory of gravity. Therefore, a unified theory of all fundamental forces remains elusive, although there are some promising candidates.

One of the first approaches to include electrodynamics into general relativity was made by Kaluza and Klein in 1921 [5] and 1926 [6], respectively. The idea was to expand spacetime from a four dimensional manifold to a five dimensional manifold, in such a way that the extra dimension is "compactified", making it unobservable to macroscopic experiments. The extra dimension in this model is just the gauge group of electrodynamics, $\mathrm{U}(1) \equiv S^1$. This approach of "gluing" a compact manifold to every spacetime point still appears in many modern theoretical models, since it allows one to consider higher dimensional spaces with additional fields while not contradicting the obvious experimental fact that the world appears to be four dimensional.

One field of study where such compactified spaces arise is string theory [7–9]. The basic idea is the formulation of a quantum theory for a one dimensional object, the string, instead of a point particle. The different observed kinds of particles should then arise as excitations of these strings. This leads to some interesting mathematical complications, one of which is that spacetime should have 10 dimensions. One way to deal with this is by considering a spacetime of the form $\mathcal{M} = \mathcal{M}_4 \times X_6$, where X_6 is a six-dimensional compact manifold, similar to the approach of Kaluza and Klein. It should be mentioned that experiments up to date failed to show any evidence of such extra dimensions, constricting the possible scale of these constructions.

This gives a physical motivation to study gauge theories on higher dimensional compact manifolds. In order to get a consistent physical theory one has to impose some conditions on the manifold describing the compact space. One approach is to demand that the manifold carries a G-Structure, meaning that the frame bundle's strucure group $\mathrm{GL}(n)$ can be reduced to a subgroup G. Such manifolds have been studied recently in the context of string theory (cf. [10–15]). Among them are Sasakian manifolds, with the prime example being the the odd dimensional spheres which can be written as $S^{2n+1} \equiv \mathrm{SU}(n+1)/\mathrm{SU}(n)$. In fact, many of these manifolds can be written as coset spaces G/H, where G is a Lie group and H is a closed subgroup such that G/H is a reductive homogenous space. They have the unifying feature that the tangent space can be decomposed into the tangent space of the subgroup H and an orthogonal subspace $\mathfrak{m}$, which may itself have preferred directions.

The goal of this thesis is to find solutions to the Yang-Mills equations on manifolds of this form, with the additional feature that the generators of the subspace $\mathfrak{m}$

can be decomposed into two distinct sets. More precisely, we will consider spaces $\mathbb{R} \times G/H$ equipped with a metric of the form $g = e^0 \otimes e^0 + g_{CK}$, where e^0 is the basis one-form in the $\mathbb{R}$ direction and g_{CK} is the Cartan-Killing metric on G/H. We will further assume that the base manifold carries a non-vanishing, totally antisymmetric torsion, and will then analyze the torsion-full Yang-Mills equations with respect to a gauge connection on the trivial principal bundle $((\mathbb{R} \times G/H) \times G), \pi, \mathbb{R} \times G/H)$. The factor $\mathbb{R}$ can either be interpreted as part of the non-compact manifold $\mathcal{M}$, or be replaced with S^1, such that $S^1 \times G/H$ is again a compact manifold; this will lead to periodic solutions. This setup encompasses many different types of manifolds, and we will construct explicit solutions for the spheres $S^{2n+1} \equiv \mathrm{SU}(n+1)/\mathrm{SU}(n)$ and $S^{4n+3} \equiv \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$, both numerically and analytically.

The first few chapters of this thesis will give a short introduction into the necessary mathematical tools to accuratly describe higher-dimensional gauge theory. Since this is a vast field, the presentation given here cannot give justice to the richness of the underlying mathematics; it is rather a reminder of the most important definitions and theorems in order to solidify the used notation and conventions. There will be some hints towards the literature giving a more thorough introduction into these topics at the beginning of each chapter.

Chapter 2 will deal with the basics of (pseudo)-Riemannian geometry. It will start with the definition of vector bundles and the construction tangent and cotangent spaces, as well as forms and give a short list of their most important properties. The aim of this chapter is the definition of a linear connection and the derived properties of torsion and curvature.
The third chapter will then give a short overview of the theory of Lie groups and Lie algebras, in the scope that is needed to describe the theory of principal fibre bundles, which is the content of chapter 4. Chapter 5 will then give some context for the field equations appearing in Yang-Mills theory, which concludes the mathematical introduction.

The rest of the thesis will then deal with the explicit construction of the cases we are interested in. In chapter 6, I will rewrite the torsion-full Yang-Mills equations into the classical equations of motion for a particle moving in a two-dimensional plain. Chapter 7 will then contain some examples where we find new, explicit solutions to the field equations.
Some information on topics that lay somewhat outside the main scope of this thesis can be found in the appendix; namely a construction of the Lie algebra of $\mathrm{Sp}(n)$, a notion on G-invariant connections, some remarks on the classical equations of motion we encountered and some numerical solutions to our final equations.

2 Differential geometry

We want to construct a field theory on a general, possibly curved space. For this, the familiar notion of a vector space is insufficient; we need a more general geometrical object on which we can perform calculations. Such a space is called a manifold, a topological space that can be locally identified with $\mathbb{R}^n$. Doing calculus on manifolds requires a considerable mathematical apparatus, some of which will be derived in this chapter.
Sources include [16–21], and omited proofs can be found therein.

2.1 (Co-)Tangent spaces and bundles

If not noted otherwise, $\mathcal{M}$ will always be a n-dimensional manifold[1] and all manifolds will be C^∞[2].

Definition 2.1.1. *A map $\gamma : I \subseteq \mathbb{R} \to \mathcal{M}$ is called a curve through the point $p \in \mathcal{M}$ if $\gamma(0) = p$.*

Two curves γ_1, γ_2 through $p \in \mathcal{M}$ are called equivalent if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ for some chart φ around p. An equivalence class of curves through a point p is called a tangent vector at p. The set of all tangent vectors at a point is called the tangent space and is denoted by $T_p\mathcal{M}$. One can write

$$Y_p \;=\; \gamma'(0) \;=\; \frac{d}{dt}\gamma(t)\bigg|_{t=0} \tag{2.1}$$

[1] A topological Hausdorff space equipped with coordinate charts $\varphi : \mathcal{U} \to \mathcal{V} \subset \mathbb{R}^n$ for every open subset $\mathcal{U} \subset \mathcal{M}$.

[2] Meaning that given two charts $\varphi_1 : \mathcal{U}_1 \to \mathcal{V}_1, \varphi_2 : \mathcal{U}_2 \to \mathcal{V}_2$, the map $\varphi_2 \circ \varphi_1^{-1}$ is C^∞ in the usual sense

for a tangent vector Y_p at the point p using a representative γ. In this Notation,

$$Y_p(f) \;=\; (f \circ \gamma)'(0) \tag{2.2}$$

is the derivative of $f \in C^\infty(\mathcal{M})$ along Y_p.

The cotangent space is the space of all linear functions $\eta : T_p\mathcal{M} \to \mathbb{R}$, the dual space to $T_p\mathcal{M}$, denoted by $T_p^*\mathcal{M}$.

Remark. The tangent space is a vector space of dimension n. A basis can be derived from the standard basis of $\mathbb{R}^n$, $\{e_i \mid i \;=\; 1, \ldots, n\}$, by setting

$$\partial_i|_p \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\varphi^{-1}\left(\varphi(p) + te_i\right)\bigg|_{t=0} \tag{2.3}$$

for some coordinate chart φ. A general tangent vector can then be expressed as

$$Y_p \;=\; a^i\partial_i\Big|_p \tag{2.4}$$

Since this definition depends on the choice of φ, one also has to require the tranformation formula

$$\tilde{a}^j \;=\; a^i \frac{\partial}{\partial x^i}\left(\tilde{\varphi}^j \circ \varphi^{-1}\right)\bigg|_{\varphi(p)} \equiv a^i \frac{\partial \tilde{x}^j}{\partial x^i} \tag{2.5}$$

for some other representation $Y_p = \tilde{a}^i\tilde{\partial}_i\Big|_p$

Definition 2.1.2. *A tripel $(\mathcal{E}, \pi, \mathcal{M})$ where $\mathcal{E}, \mathcal{M}$ are manifolds and $\pi : \mathcal{E} \to \mathcal{M}$ is a map is called a $\mathbb{K}$-vector bundle if*

(i) the fibre $\mathcal{E}_p := \pi^{-1}(p)$ is a n-dimensional vector space for every $p \in \mathcal{M}$

(ii) for every $p \in \mathcal{M}$ exists an open neighbourhood $\mathcal{U} \subseteq \mathcal{M}$ around p and a diffeomorphism

$$\phi \;=\; \left(\phi^1, \phi^2\right) : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{K} \tag{2.6}$$

such that $\phi^1 = \pi|_{\pi^{-1}(\mathcal{U})}$ and that $\phi^2_{\mathcal{E}_q} : \mathcal{E}_q \to \mathbb{K}^n$ is $\mathbb{K}$-linear for every $q \in \mathcal{M}$

Example 2.1.3. Both $T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$ and $T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p^*\mathcal{M}$ are vector bundles, as well as the tensor bundle

$$T_q^p\mathcal{M} := \underbrace{T\mathcal{M} \otimes \cdots \otimes T\mathcal{M}}_{ptimes} \otimes \underbrace{T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M}}_{qtimes} \tag{2.7}$$

Definition 2.1.4. *A smooth map $\sigma : \mathcal{U} \subseteq \mathcal{M} \to \mathcal{E}$ is called a section of a vector bundle $(\mathcal{E}, \pi, \mathcal{M})$ if $\pi(\sigma(p)) = p$ for all $p \in \mathcal{U}$. σ is called a global section if $\mathcal{U} = \mathcal{M}$, otherwise it is called a local section. The set of global sections of $\mathcal{E}$ is denoted by $\Gamma(\mathcal{E})$.*

Example 2.1.5. The global sections of the tangent space $\Gamma(T\mathcal{M}) \equiv \mathfrak{X}(\mathcal{M})$ are called vector fields. Similary to (2.4), any vector field can be written as $Y_p = a^i(p)\partial_i|_p$, now with $a^i \in C^\infty(\mathcal{M})$. Equation (2.2) now defines a C^∞-map $p \mapsto Y_p(f)$, simply denoted by $Y(f)$.

For two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, one can define the commutator $[X, Y] \in \mathfrak{X}(\mathcal{M})$ as

$$[X, Y]_p(f) \;=\; X_p(Y(f)) - Y_p(X(f)). \tag{2.8}$$

It is antisymmetric and obeys the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \;=\; 0. \tag{2.9}$$

Similary, the global sections of the cotangent space $\Gamma(T^*\mathcal{M}) \equiv \Omega^1(\mathcal{M})$ are called one-forms. They can be written as

$$\eta_p \;=\; \eta_k(p)\, \mathrm{d}x^k\Big|_p \tag{2.10}$$

where $\mathrm{d}x^k$ is the dual basis to ∂_i, obeying $\mathrm{d}x^k(\partial_i) = \delta^k_i$. If $Y \in \mathfrak{X}(\mathcal{M})$ with $Y_p = y^i(p)\, \partial_i|_p$, then

$$\begin{aligned}(\eta(Y))\,(p) &= \eta_k(p)\, \mathrm{d}x^k\Big|_p \left(y^i(p)\, \partial_i|_p\right) \\ &= \eta_k(p) y^k(p)\end{aligned} \tag{2.11}$$

General tensor fields are evaluated analogously: $T \in T^p_q\mathcal{M}$ given by

$$T|_p \;=\; T^{b_1,\dots,b_p}_{a_1,\dots,a_q}(p)\, \partial_{b_1}|_p \otimes \cdots \otimes \partial_{b_p}\Big|_p \otimes \mathrm{d}x^{a_1}|_p \otimes \cdots \otimes \mathrm{d}x^{a_q}|_p \tag{2.12}$$

results in

$$\Big(T(\eta^1, \dots, \eta^p, X_1, \dots, X_q)\Big)\,(p) = T^{b_1,\dots,b_p}_{a_1,\dots,a_q}(p)\eta^1_{b_1}(p)\dots\eta^p_{b_p}(p)X^{a_1}_1(p)\dots X^{a_q}_q(p). \tag{2.13}$$

Remark. The coordinate basis fields ∂_i used above obey the commutation relations

$$[\partial_i, \partial_j] \;=\; 0. \tag{2.14}$$

One could also choose a more general basis $\{E_i \mid i \;=\; 1\dots, n\}$, satisfying

$$[E_i, E_j] \;=\; f_{ij}^k E_k \tag{2.15}$$

for $T_p\mathcal{M}$ (and $\{e^j \mid j \;=\; 1\ldots,n\}$ defined by $e^j(E_i) = \delta_i^j$ for $T_p^*\mathcal{M}$, respectively).

In this case, the Jacobi identity becomes a condition for the structure constants by applying it to $X, Y, Z = E_i, E_j, E_k$. It then reads

$$f_{ij}^l f_{kl}^m + f_{ki}^l f_{jl}^m + f_{jk}^l f_{il}^m \;=\; 0 \tag{2.16}$$

for all $i, j, k, m \in 1, \ldots, \dim\mathcal{M}$.
This will prove useful later when we consider group manifolds.

2.2 k-forms and derivatives

Definition 2.2.1. *A k-form on $\mathcal{M}$ is a tensor field $\eta \in T_k^0\mathcal{M}$ such that η_p takes value in the antisymmetric tensor product $\Lambda^k(T_p^*\mathcal{M})$ for every $p \in \mathcal{M}$.*
The space of k-forms is denoted by $\Omega^k(\mathcal{M})$, where $\Omega^0(\mathcal{M})$ is identified with the space of C^∞-functions on $\mathcal{M}$. If $\eta \in \Omega^k(\mathcal{M})$ and $\omega \in \Omega^l(\mathcal{M})$ then we can define the wedge product $\eta \wedge \omega \in \Omega^{k+l}(\mathcal{M})$ by

$$\eta \wedge \omega(X_1, \ldots, X_{k+l}) = \frac{1}{k!l!}\sum_\sigma (-1)^\sigma \eta(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\omega(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}). \tag{2.17}$$

Here, $\sum_\sigma$ is the sum over all possible permutations σ.

Any k-form η can be written locally as

$$\eta \;=\; \frac{1}{k!}\eta_{a_1,\ldots,a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k} \tag{2.18}$$

where $\eta_{a_1,\ldots,a_k} = \eta(\partial_{a_1}, \ldots, \partial_{a_k})$

If $(\mathcal{E}, \pi, \mathcal{M})$ is some vector bundle, a section $\eta \in \Gamma(\mathcal{E} \otimes \Lambda^k(T^*\mathcal{M})) \equiv \Omega^k(\mathcal{M}, \mathcal{E})$ is called a k-form with values in $\mathcal{E}$. For example, a tensor $T \in T_k^1\mathcal{M}$ that is antisymmetric in the lower indices can be interpreted as a k-form with values in $T\mathcal{M}$. One can always write $\eta = \xi^i \otimes \eta_i$ for such a form (using suitable $\xi \in \Gamma(\mathcal{E})$, $\eta_i \in \Omega^k(\mathcal{M})$),meaning that

$$\eta(X_1, \ldots, X_k) \;=\; \xi^i \eta_i(X_1, \ldots, X_k) \tag{2.19}$$

Definition 2.2.2. *The exterior derivative can be defined as the unique map* $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ *satisfying*

(i) $df(X) \;=\; X(f)$

(ii) $d(df) \;=\; 0$

(iii) $d(\eta + \eta') \;=\; d\eta + d\eta'$

(iv) $d(\eta \wedge \omega) \;=\; d\eta \wedge \omega + (-1)^k \eta \wedge d\omega$

where $f \in \Omega^0(\mathcal{M})$, $\eta, \eta' \in \Omega^k(\mathcal{M})$, $\omega \in \Omega^l(\mathcal{M})$ *and* $X \in \mathfrak{X}(\mathcal{M})$.

Lemma 2.2.3. *In local coordinates, the exterior derivative of a k-form* $\eta = \frac{1}{k!}\eta_{a_1,\dots,a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k}$ *is given by*

$$\begin{aligned} d\eta &= \frac{1}{k!} d(\eta_{a_1,\dots,a_k}) \wedge dx^{a_1} \wedge \cdots \wedge dx^{a_k} \\ &= \frac{1}{k!} \partial_i(\eta_{a_1,\dots,a_k}) dx^i \wedge dx^{a_1} \wedge \cdots \wedge dx^{a_k} \end{aligned} \tag{2.20}$$

It satisfies the general relation

$$\begin{aligned} d\eta(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i \left(\eta(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_{k+1})\right) \\ + \sum_{1 \le i < j \le n} &(-1)^{i+j} \eta\left([X_i, X_j], X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_{k+1}\right) \end{aligned} \tag{2.21}$$

or more concisely for $\eta \in \Omega^1(\mathcal{M})$,

$$d\eta(X, Y) \;=\; X(\eta(Y)) - Y(\eta(X)) - \eta\left([X, Y]\right). \tag{2.22}$$

Remark. Equations (2.20) and (2.21) are each equivalent to the definition 2.2.2.

Definition 2.2.4. *If* $f : \mathcal{M} \to \mathcal{N}$ *is a map than the differential (or push forward) of that map* $f_{*p} : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ *is defined by*

$$f_{*p}(Y_p) \;=\; (f \circ \gamma)'(0) \tag{2.23}$$

where γ *is a representative of the vector* Y_p.

Analogously, the pull-back $f^*\eta \in \Omega^k(\mathcal{N})$ *of a k-form* $\eta \in \Omega^k(\mathcal{M})$ *is defined by*

$$(f^*\eta)_p(X_1, \dots, X_k) \;=\; \eta_{f(p)}(f_{*p}X_1, \dots, f_{*p}X_k), \tag{2.24}$$

for $X_1, \dots, X_k \in T_p\mathcal{M}$. *The pull-back has the following properties:*

(i) $df^*\eta \;=\; f^*d\eta$

(ii) $f^*(\eta \wedge \omega) \;=\; f^*\eta \wedge f^*\omega$

(iii) $(f \circ g)^*\eta \;=\; g^*f^*\eta$

If $f : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, one can also define the pull-back of a vector field X by

$$f^*X \;=\; f_*^{-1}X, \tag{2.25}$$

or, if T is a mixed tensor of vector and co-vectors, by applying the appropriate definition for each factor in the tensor product; e.g., if $T = \eta \otimes X$, then

$$f^*T \;=\; f^*\eta \otimes f_*^{-1}X \tag{2.26}$$

Lemma 2.2.5. *Suppose $\{E_a \mid a = 1, \ldots, n\}$ is a general basis of the tangent bundle and $\{e^a \mid a = 1, \ldots, n\}$ is the dual basis, satisfying $E_a(e^b) = \delta_a^b$ and $[E_a, E_b] = f_{ab}^c E_c$. Then the exterior derivative of the dual basis satisfies*

$$de^a \;=\; -\tfrac{1}{2} f_{bc}^a e^b \wedge e^c \tag{2.27}$$

This is known as the Maurer-Cartan equation.

Remark. This is a simple application of the general formula (2.22) for $\eta = e^a$, $X = E_b$, $Y = E_c$:

$$\begin{aligned}
\mathrm{d}e^a(E_b, E_c) &= \underbrace{E_b \overbrace{e^a(E_c)}^{=\delta_c^a}}_{=0} - \underbrace{E_c \overbrace{e^a(E_b)}^{=\delta_b^A}}_{=0} - e^a([E_b, E_c]) \\
&= -\, e^a(f_{bc}^d E_d) \\
&= -\, f_{bc}^d e^a(E_d) \;=\; -f_{bc}^a \\
&= -\, \tfrac{1}{2}\,(f_{bc}^a - f_{cb}^a)
\end{aligned} \tag{2.28}$$

or in other words

$$\mathrm{d}e^a \;=\; -\tfrac{1}{2} f_{bc}^a e^b \wedge e^c \tag{2.29}$$

Definition 2.2.6. *Let $Y \in \mathfrak{X}(\mathcal{M})$ be a vector field such that for every $p \in \mathcal{M}$ there exists a curve $\gamma_p : \mathbb{R} \to \mathcal{M}$ through p with $\gamma_p'(t) = Y_{\gamma_p(t)}$ for all $t \in \mathbb{R}$ (i.e. Y is a complete vector field). Then the set*

$$\{\varphi_t : \mathcal{M} \to \mathcal{M} \mid \varphi_t(p) \;=\; \gamma_p(t)\ \forall t \in \mathbb{R}\} \tag{2.30}$$

is called the one-parameter group generated by Y. φ_t is a diffeomorphism, and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$

Definition 2.2.7. *The Lie derivative is the unique map $L : \mathfrak{X}(\mathcal{M}) \times T^l_m\mathcal{M} \to T^l_m\mathcal{M}$ satisfying*

(i) $L_X f \;=\; X(f)$

(ii) $L_X Y \;=\; [X, Y]$

(iii) the map $T \mapsto L_X T$ *is* $\mathbb{R}$*-linear*

(iv) $L_X(T \otimes T') \;=\; L_X T \otimes T' + T \otimes L_X T'$

(v) $C \circ L_X \;=\; L_X \circ C$

for all $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$ and all $T \in T^l_m\mathcal{M}, T' \in T^{l'}_{m'}\mathcal{M}$. Here C denotes the contraction map, $C(\eta \otimes X) = \imath_X\eta = \eta(X)$ (see definition 2.3.5).

Remark. This actually defines the Lie derivative for arbitrary tensor fields. For example, one can calculate that the Lie derivative of a k-form η is given by

$$L_X\eta \;=\; (\mathrm{d} \circ \imath_X + \imath_X \circ \mathrm{d})\eta. \tag{2.31}$$

Lemma 2.2.8. *The Lie derivative of a tensor field T along a vector Y can also be defined by*

$$L_Y T \;=\; \frac{d}{dt}\varphi_t^* T\bigg|_{t\,=\,0} \tag{2.32}$$

where $\{\varphi_t\}$ is the one-parameter group generated by Y.

2.3 (Pseudo-)Riemannian metrics

Definition 2.3.1. *A pseudo-Riemannian metric is a tensor $g \in T^0_2\mathcal{M}$ such that for all $p \in \mathcal{M}$, g_p is a smooth map satisfying*

(i) $g_p(X, Y) \;=\; g_p(Y, X)$ *(symmetric)*

(ii) $g_p(aX + Y, Z) \;=\; a g_p(X, Z) + g_p(Y, Z)$ *(linear)*

(iii) $g_p(X, Y) \;=\; 0 \;\forall Y \in T_p\mathcal{M} \implies X \;=\; 0$ *(non-degenerate)*

The signature (s_+, s_-) of a metric is defined as the maximal dimension of the linear subspaces of $T_p\mathcal{M}$ such that g_p constrained to the subspace is positive (s_+) or negative (s_-) definite. g is called a Riemanian metric if it has signature $(dim(\mathcal{M}), 0)$ and a Lorentzian metric if it has signature $(dim(\mathcal{M}) - 1, 1)$ or $(1, dim(\mathcal{M}) - 1)$. Similary, a manifold is $\mathcal{M}$ is called Riemannian (Lorentzian) if it carries a Riemanian (Lorentzian) metric.

A metric can also be interpreted as an inner product for every point of the manifold by setting

$$\langle X, Y\rangle\big|_p := g\big|_p (X, Y), \tag{2.33}$$

with $X, Y \in T_p\mathcal{M}$.

Remark. Given a metric g and a vector $X \in \mathfrak{X}(\mathcal{M})$, one can define the covariant vector $X^\flat \in \Omega^1(\mathcal{M})$ by

$$X^\flat \;=\; g(X, \cdot) \tag{2.34}$$

If $\{e^i\}$ is a local basis for the cotangent space, this reads

$$X^\flat \;=\; {X^\flat}_i e^i \;=\; g_{ij}X^j e^i \Longleftrightarrow {X^\flat}_i \;=\; g_{ij}X^j, \tag{2.35}$$

meaning that one can use the metric to pull down the indices of any vector (or, more generally, any tensor in $T^l_0\mathcal{M}$). Additionally, one can define the dual metric as the $g^{-1} \in T^2_0\mathcal{M}$ satisfying

$$g_{ab}(g^{-1})^{bc} \;=\; \delta^c_a. \tag{2.36}$$

Using this, one can pull up co-vector indices (often denoted by $\eta^\sharp$). In conjunction with (2.35), this allows us to pull indices of general tensors $T \in T^l_m\mathcal{M}$ up and down. Note that if we use g^{-1} to pull up the indices of g, we just get g^{-1} again, hence we will use the symbol g for the metric as well as the dual metric.

We will not distinguish between X and $X^\flat$ or η and $\eta^\sharp$ if we calculate something using a local basis, since the index positions already indicates if we look at the vector or the corresponding covector. Note that we will do most calculations using $g_{ab} = \delta_{ab}$; in this case, one can just write $X^a = X_a$, implying that this is an equation for the components of the vector X and the co-vector $X^\flat$ (or, analogously, for general tensors).

Definition 2.3.2. *We can extend the inner product of vectors and one-forms defined by g and g^{-1} to an inner product for tensor product spaces by taking the product slotwise. If for example $\eta, \mu \in T^0_k\mathcal{M}$, we can write*

$$\langle \eta, \mu\rangle = \left\langle \eta^1 \otimes \cdots \otimes \eta^k, \mu^1 \otimes \cdots \otimes \mu^k \right\rangle := \prod_{a=1}^{k} g^{-1}(\eta^a, \mu^a). \tag{2.37}$$

If $\{e^a\}$ is a basis of $T^*\mathcal{M}$, we can write $\eta = \eta_{a_1\ldots a_k} e^{a_1}\otimes\cdots\otimes e^{a_k}$ and $\mu = \mu_{a_1\ldots a_k} e^{a_1}\otimes \cdots \otimes e^{a_k}$; The inner product becomes

$$\langle \eta_{a_1\ldots a_k} e^{a_1} \otimes \cdots \otimes e^{a_k}, \mu_{a_1\ldots a_k} e^{a_1} \otimes \cdots \otimes e^{a_k}\rangle \;=\; \eta_{a_1\ldots a_k}\mu^{a_1\ldots a_k}. \tag{2.38}$$

In the case where $\eta, \mu \in \Omega^k(\mathcal{M})$, *we get*

$$\langle \eta, \mu \rangle \;=\; \left\langle \eta^1 \wedge \cdots \wedge \eta^k, \mu^1 \wedge \cdots \wedge \mu^k \right\rangle := k! \sum_\sigma (-1)^\sigma \prod_{a=1}^{k} g^{-1}(\eta^a, \mu^{\sigma(a)}), \quad (2.39)$$

and hence

$$\left\langle \tfrac{1}{k!}\eta_{a_1 \dots a_k} e^{a_1} \wedge \cdots \wedge e^{a_k}, \tfrac{1}{k!}\mu_{a_1 \dots a_k} e^{a_1} \wedge \cdots \wedge e^{a_k} \right\rangle \;=\; \eta_{a_1 \dots a_k}\mu^{a_1 \dots a_k}. \quad (2.40)$$

When dealing with k-forms, we will use a renormalized inner product by setting

$$\langle \eta, \mu \rangle_\wedge = \tfrac{1}{k!} \langle \eta, \mu \rangle, \quad (2.41)$$

such that

$$\left\langle \tfrac{1}{k!}\eta_{a_1 \dots a_k} e^{a_1} \wedge \cdots \wedge e^{a_k}, \tfrac{1}{k!}\mu_{a_1 \dots a_k} e^{a_1} \wedge \cdots \wedge e^{a_k} \right\rangle_\wedge \;=\; \tfrac{1}{k!}\eta_{a_1 \dots a_k}\mu^{a_1 \dots a_k}. \quad (2.42)$$

Definition 2.3.3. *A (pseudo)-Riemannian n-dimensional manifold* $\mathcal{M}$ *is called orientable if there exists a nowhere vanishing n-form* ε, *called the volume form. Given an orthonormal basis* $\{e^a\}$ *(with respect to a metric g), we can write*

$$\varepsilon := e^1 \wedge \cdots \wedge e^n. \quad (2.43)$$

The Hodge-duality map $* : \Omega^k(\mathcal{M}) \to \Omega^{n-k}(\mathcal{M})$ *(also called the Hodge star operator) is then defined by*

$$\eta \wedge *\mu = \langle \eta, \mu \rangle_\wedge \varepsilon, \quad (2.44)$$

where $\eta, \mu \in \Omega^k(\mathcal{M})$

Lemma 2.3.4. *The Hodge dual has the following useful properties: Given a basis* $\{E_a\}$ *of* $T\mathcal{M}$ *and its dual basis* $\{e^a\}$ *of* $T^*\mathcal{M}$, *the Hodge star satisfies*

$$*\left(e^{a_1} \wedge \cdots \wedge e^{a_k}\right) \;=\; \tfrac{1}{(n-k)!}\varepsilon^{a_1 \dots a_k}{}_{a_{k+1} \dots a_n} e^{a_{k+1}} \wedge \cdots \wedge e^{a_n}. \quad (2.45)$$

This implies that for a k-form

$$\eta = \tfrac{1}{k!}\eta_{a_1 \dots a_k} e^{a_1} \wedge \cdots \wedge e^{a_k} \quad (2.46)$$

we get

$$*\,\eta \;=\; \tfrac{1}{(n-k)!}(*\eta)_{b_1 \dots b_{n-k}} e^{b_1} \wedge \cdots \wedge e^{b_{n-k}}, \quad (2.47)$$

with

$$(*\eta)_{b_1 \dots b_{n-k}} \;=\; \tfrac{1}{k!}\eta_{a_1 \dots a_k}\varepsilon^{a_1 \dots a_k}{}_{b_1 \dots b_{n-k}}. \quad (2.48)$$

Furthermore, we have

(i) $*\varepsilon = 1$

(ii) $*(\eta \wedge *\mu) = \langle \eta, \mu \rangle_\wedge$

(iii) $* * \eta = (-1)^{k(n-k)+s_-} \eta$

(iv) $\eta \wedge *\mu = \mu \wedge *\eta = (-1)^{k(n-k)} * \eta \wedge \mu$

(v) $\langle *\eta, *\mu \rangle_\wedge = (-1)^{s_-} \langle \eta, \mu \rangle_\wedge$

for $\eta, \mu \in \Omega^k(\mathcal{M})$, and s_- defined as in 2.3.1.

Definition 2.3.5. *Given a vector $X \in \mathfrak{X}(\mathcal{M})$ and a k-form $\eta \in \Omega^k(\mathcal{M})$ the contraction of X and η is defined by*

$$(X \lrcorner \eta)(X_1, \ldots, X_{k-1}) = \eta(X, X_1, \ldots, X_{k-1}). \tag{2.49}$$

Notice that the position where we insert X into η only gives a different sign, since η is totally antisymmetric. The contraction can also be defined as a map $\imath_X : \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ or as a map $C : \Omega^k(\mathcal{M}, \mathfrak{X}(\mathcal{M})) \to \Omega^{k-1}(\mathcal{M})$ by setting

$$(X \lrcorner \eta) \equiv \imath_X \eta \equiv C(X \otimes \eta). \tag{2.50}$$

Similary, given a tensor $T \in T^p_q\mathcal{M}$ $(p, q \neq 0)$, one can define the contracted tensor $\tilde{T} \in T^{p-1}_{q-1}\mathcal{M}$ by setting

$$(\tilde{T})^{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_p}_{b_1,\ldots,b_{j-1},b_{j+1},\ldots,b_q} = T^{a_1,\ldots,a_{i-1},n,a_{i+1},\ldots,a_p}_{b_1,\ldots,b_{j-1},n,b_{j+1},\ldots,b_q} \tag{2.51}$$

with a sum over the index n. In this case we have to specify which index gets contracted, because we did not assume T has any symmetries.

Given a metric, we can use it to contract a purely co- or contravariant tensor. If for example $\eta = \eta_{ab} e^a \otimes e^b$, we can write $\tilde{\eta} = \eta^a_a = g^{ab}\eta_{ab} \in C^\infty(\mathcal{M})$.

Remark. Using the Hodge star, we have

$$X \lrcorner * \eta = *(\eta \wedge X^\flat). \tag{2.52}$$

In fact, one can use the Hodge operator to define the contraction of a p-form η with a q-form μ (for $p \leq q$) by setting

$$\eta \lrcorner \mu := *(\eta \wedge *\mu). \tag{2.53}$$

For $p = q$, this is just the inner product defined in 2.3.2.

2.4 Covariant derivatives, torsion and curvature

Definition 2.4.1. *Let $(\mathcal{E}, \pi, \mathcal{M})$ be a vector bundle over $\mathcal{M}$. A linear connection is a map $\nabla : \Gamma(\mathcal{E}) \to \Omega^1(\mathcal{M}, \mathcal{E})$ such that*

(i) $\forall\ \xi, \xi' \in \Gamma(\mathcal{E}) :\ \nabla(\xi + \xi') \ =\ \nabla\xi + \nabla\xi'$

(ii) $\forall\ \xi \in \Gamma(\mathcal{E}), f \in C^\infty(\mathcal{M}) :\ \nabla(f\xi) \ =\ \xi \otimes df + f\nabla\xi$

For $Y \in \mathfrak{X}(\mathcal{M})$, $\nabla_Y\xi \equiv (\nabla\xi)(Y) \in \Gamma(\mathcal{E})$ is called the covariant derivative of ξ along Y. Condition (ii) then reads

$$\nabla_Y(f\xi) \ =\ Y(f)\xi + \nabla_Y\xi \tag{2.54}$$

If $\{E_a \mid a \ =\ 1, \ldots, n\}$ is a local basis for $T\mathcal{M}$ then the covariant derivative of one of the basis vectors along another satisfies

$$\nabla_{E_a}E_b \ =\ \omega^c_{ab}E_c, \tag{2.55}$$

where $\omega^c_{ab} \in C^\infty(\mathcal{M})$ are called the connection components. Using these, the covariant derivative of an associated co-basis vector e^b is

$$\nabla_{E_a}e^b \ =\ -\omega^b_{ac}e^c \tag{2.56}$$

The covariant derivative of an arbitrary tensor in this coordinate basis is then given by

$$\begin{aligned}\nabla_{E_c}T \ &= \nabla_{E_c}\left(T^{b_1,\ldots,b_p}_{a_1,\ldots,a_q}E_{b_1} \otimes \cdots \otimes E_{b_p} \otimes e^{a_1} \otimes \cdots \otimes e^{a_q}\right)\\ &= \left(\nabla_{E_c}T^{b_1,\ldots,b_p}_{a_1,\ldots,a_q}\right)E_{b_1} \otimes \cdots \otimes E_{b_p} \otimes e^{a_1} \otimes \cdots \otimes e^{a_q},\end{aligned} \tag{2.57}$$

with

$$\begin{aligned}\nabla_{E_c}T^{b_1,\ldots,b_p}_{a_1,\ldots,a_q} \ =\ &\partial_c T^{b_1,\ldots,b_p}_{a_1,\ldots,a_q} + \omega^{b_1}_{cd}T^{d,b_2,\ldots,b_p}_{a_1,\ldots,a_q} + \cdots + \omega^{b_p}_{cd}T^{b_1,\ldots,b_{p-1},d}_{a_1,\ldots,a_q}\\ &- \omega^d_{ca_1}T^{b_1,\ldots,b_p}_{d,a_2,\ldots,a_q} + \cdots + \omega^d_{ca_q}T^{b_1,\ldots,b_p}_{a_1,\ldots,a_{q-1},d}.\end{aligned} \tag{2.58}$$

Remark. The connection components can be interpreted as an endomorphism valued one-form:

$$\begin{aligned}\omega \ &= \omega^c_{ab}e^a \otimes (E_c \otimes e^b)\\ &\equiv \omega^c{}_b E_c \otimes e^b\end{aligned} \tag{2.59}$$

Note that this is not a tensor since it depends on the choice of basis. If we were to chose a different basis satisfying $\tilde{E}_a = f^b_a E_b$ and calculated the connection components, we would get

$$\tilde{\omega}^c_{ab} \ =\ (f^{-1})^c_k\omega^k_{ij}f^i_a f^j_b + (f^{-1})^c_k\tilde{E}_a(f^k_b). \tag{2.60}$$

The first term is what one would expect if ω was in fact a tensor (i.e., the object ω written in different coordinates). However, we get an additional term, dependent on the transformation f. If we interpreted f as an endomorphism valued local function, we could write

$$\tilde{\omega} \;=\; \omega + f^{-1}\mathrm{d}f \tag{2.61}$$

for the new connection one-form, with respect to the new basis. We will encounter this formula again, when we define the more general notion of a connection on a principal bundle in chapter 4.

The exterior derivative of the connection one-form can be calculated using equation (2.22):

$$(\mathrm{d}\omega^a{}_b)_{cd} \;=\; E_c(\omega^a_{db}) - E_d(\omega^a_{cb}) - f^n_{cd}\omega^a_{nb} \tag{2.62}$$

Definition 2.4.2. *The torsion of a linear connection ∇ is the tensor $T \in T^1_2\mathcal{M}$ defined by*

$$T(X,Y) \;=\; \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.63}$$

It is antisymmetric in the lower indices and, given a basis $\{E_a\}$, $\{e^b\}$, can be expressed as

$$\begin{aligned} T &= T^a_{bc}E_a \otimes e^b \otimes e^c \\ &= \tfrac{1}{2}T^a_{bc}E_a \otimes e^b \wedge e^c \end{aligned} \tag{2.64}$$

which means that the torsion can be interpreted as a vector valued two-form. Since the E_a obey the commutation relation $[E_a, E_b] = f^c_{ab}E_c$, the torsion components can be written using the connection components (2.55):

$$T^c_{ab} \;=\; \omega^c_{ab} - \omega^c_{ba} - f^c_{ab} \tag{2.65}$$

$$\Longleftrightarrow \qquad T^c \;=\; (\omega^c_{ab} - \tfrac{1}{2}f^c_{ab})e^a \wedge e^b \tag{2.66}$$

Using the Maurer-Cartan equations 2.2.5, this can be rewritten into the first Cartan structure equation:

$$T^c \;=\; de^c + \omega^c{}_b \wedge e^b \tag{2.67}$$

Given a metric g, we can define the covariant torsion tensor

$$\Theta(X,Y,Z) \;=\; g\left(X, T(Y,Z)\right) \tag{2.68}$$

with the components

$$\Theta_{abc} \;=\; g_{cd}T^d_{ab}. \tag{2.69}$$

This is again antisymmetric in the first two indices, but not necessarily totally antisymmetric.

Example 2.4.3. There is a unique linear connection on any Riemanian manifold, called the Levi-Civita connection. It satisfies

(i) $\nabla_X g \;=\; 0 \;\forall X \in \mathfrak{X}(\mathcal{M})$ (metric)

(ii) $T \equiv 0$ (torsion free)

Its components with respect to a given basis $\{E_a\}$ satisfying $[E_a, E_b] = f^c_{ab} E_c$ can be calculated out of the metric coefficients:

$$\omega^c_{ab} \;=\; \tfrac{1}{2} g^{cd} \left(-E_d g_{ab} + E_a g_{bd} + E_b g_{da}\right) + \tfrac{1}{2}\left(f^c_{ab} + g^{ce} f^d_{ea} g_{bd} + g^{ce} f^d_{eb} g_{da}\right) \tag{2.70}$$

If we are given another metric connection which has non vanishing torsion, it will describe the same geodesics (i.e., curves γ that satisfy $\nabla_{\dot\gamma}\dot\gamma = 0$) as the Levi-Civita connection if and only if the torsion tensor Θ is totally antisymmetric.

Remark. The notion of a geodesic curve is of great importance in general relativity. $\nabla_{\dot\gamma}\dot\gamma = 0$ means that the "velocity" of the curve is constant with respect to the change of coordinates along the way. This in turn means that geodesics are the equivalent of straight lines in curved spacetime, and they describe the motion of non-accelerated observers in general relativity.

Definition 2.4.4. *The (Riemannian) curvature of a linear connection ∇ is a tensor field $R \in T^1_3\mathcal{M}$ defined by*

$$R(X,Y)Z \;=\; \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{2.71}$$

Its components with respect to a basis $\{E_a\}$ are given by

$$R^a{}_{bcd} \;=\; E_c(\omega^a_{db}) + \omega^a_{cn}\omega^n_{db} - E_d(\omega^a_{cb}) - \omega^a_{dn}\omega^n_{cb} - f^n_{cd}\omega^a_{nb}. \tag{2.72}$$

where

$$R(E_c, E_d)E_b \;=\; R^a{}_{bcd} E_a \tag{2.73}$$

Alternatively, the curvature can be interpreted as an endomorphism valued two-form

$$R(X,Y) \;=\; \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \qquad R^a{}_b \;=\; \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d \tag{2.74}$$

since it is always antisymmetric in the first two arguments. Its components can then be determined from (2.72) by using equation (2.62):

$$(R^a{}_b)_{cd} \;=\; R^a{}_{bcd} \;=\; \underbrace{E_c(\omega^a_{db}) - E_d(\omega^a_{cb}) - f^n_{cd}\omega^a_{nb}}_{=(d\omega^a{}_b)_{cd}} + \omega^a_{cn}\omega^n_{db} - \omega^a_{dn}\omega^n_{cb} \tag{2.75}$$

$$\iff R^a{}_b \;=\; d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{2.76}$$

$$\iff R \;=\; d\omega + \omega \wedge \omega \tag{2.77}$$

This last equation is known as the second Cartan structure equation.

Example 2.4.5. Let $(\mathcal{M}, g)$ be a (pseudo-)Riemannian manifold and let $R^a{}_{bcd}$ be the components of the curvature of the Levi-Civita connection. The Ricci-Tensor is defined by

$$\mathrm{Ric}_{ab} \;=\; R^c{}_{acb}. \tag{2.78}$$

The manifold is then called Einstein if the Ricci tensor is proportional to the metric (i.e., $\mathrm{Ric}_{ab} = \varkappa g_{ab}$). These manifolds play a special role in general relativity; the Einstein field equations are given by

$$\mathrm{Ric}_{ab} - g_{ab}(\tfrac{1}{2}R + \Lambda) \;=\; 8\pi T_{ab}, \tag{2.79}$$

where $R = \mathrm{Ric}^a_a$ is the Ricci scalar, Λ is the cosmological constant and T is the stress-energy tensor. Saying that a manifold is Einstein then means that the metric solves the vacuum (i.e., $T = 0$) Einstein field equations.

3

Lie groups and algebras

In the previous chapter, we interpreted connections as endomorphism valued forms that have a certain behaviour under coordinate transformations. This idea can be generalized; for this, it has to be understood that the endomorphisms of a vector space (i.e., the set of all matrices) can be identified with the infinitesimal action of the general linear group $\mathrm{GL}(V)$ of that vector space. This can be made precise by interpreting $\mathrm{GL}(V)$ as a manifold where the group multiplication is a differentiable map; such a manifold is called a Lie group. The tangent space of this manifold, called the Lie algebra, is then given by the set of endomorphisms. Given this structure, one can define a more general notion of a connection as a one-form taking values in the Lie algebra of an arbitrary group.

The actual definition of a general connection will have to wait till chapter 4, because we first have to define what exactly Lie groups and algebras are. I will follow the presentation in [16, 20, 21] to give a short overview of the most relevant definitions and properties that we will need later on. A more complete treatise of this topic can be found in [22].

3.1 Groups and actions

Definition 3.1.1. *A group is a set G equipped with a group multiplication $G \times G \to G$, $(g, h) \mapsto gh$, satisfiying*

(i) $a, b \in G \implies ab \in G$ *(closure)*

(ii) $\forall a, b, c \in G : (ab)c = a(bc)$ *(associativity)*

(iii) $\exists \mathbb{1} \in G\ \forall a \in G : \mathbb{1}a = a\mathbb{1} = a$ *(identity element)*

(iv) $\forall a \in G\ \exists a^{-1} \in G : aa^{-1} = a^{-1}a = \mathbb{1}$ *(inverse element)*

If the multiplication is commutative as well (i.e., $ab = ba\ \forall a, b \in G$), the group is called commutative or abelian.

Example 3.1.2. Probably the most important example of groups are matrix groups, subgroups of the general linear group

$$\mathrm{GL}(n, \mathbb{K}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid \det A \neq 0\} \tag{3.1}$$

using the standard matrix multiplication. The ones most frequently encountered are

$$\mathrm{SL}(n, \mathbb{K}) = \{A \in \mathrm{GL}(n, \mathbb{K}) \mid \det A = 1\} \tag{3.2}$$
$$\mathrm{SO}(n) = \left\{A \in \mathrm{SL}(n, \mathbb{R}) \mid AA^T = \mathbb{1}_n\right\} \tag{3.3}$$
$$\mathrm{U}(n) = \left\{A \in \mathrm{GL}(n, \mathbb{C}) \mid AA^\dagger = \mathbb{1}_n\right\} \tag{3.4}$$
$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\} \tag{3.5}$$
$$\mathrm{Sp}(n) = \left\{A \in \mathrm{GL}(n, \mathbb{H}) \mid AA^\dagger = 1\right\} \tag{3.6}$$

where $\mathbb{H}$ are the quaternions. For more information on $\mathrm{Sp}(n)$, see appendix A.

Definition 3.1.3. *A Lie group G is a C^∞-manifold that is also a group, in such a way that both the group multiplication and the map $g \to g^{-1}$ are C^∞ maps. We say that a Lie group G acts on a manifold $\mathcal{M}$ from the left if there exists a smooth map $\Phi : G \times \mathcal{M} \to \mathcal{M}$ with*

(i) $\forall g, g' \in G, x \in \mathcal{M} : \Phi(gg', x) = \Phi(g, \Phi(g', x))$
$$\iff (gg')x = g(g'x)$$
$$\iff L_{gg'} = L_g \circ L_{g'}$$

(ii) $\forall x \in \mathcal{M} : \Phi(\mathbb{1}, x) = x$
$$\iff \mathbb{1}x = x$$
$$\iff L_{\mathbb{1}} = \mathrm{id}_{\mathcal{M}}$$

using the shorthand $\Phi(g, x) \equiv gx$ and the left translation $L_g : \mathcal{M} \to \mathcal{M}$ defined by $L_g(x) = gx$, respectively. Note that L_g is a diffeomorphism and that $L_g^{-1} = L_{g^{-1}}$.

Analogously, one can define the right action of G on $\mathcal{M}$ as a map $\Psi : \mathcal{M} \times G \to \mathcal{M}$ satisfiying

(i) $\forall g, g' \in G, x \in \mathcal{M} : \Psi(x, gg') = \Psi(\Psi(x, g'), g)$
$$\iff x(gg') = (xg)g'$$
$$\iff R_{gg'} = R_{g'} \circ R_g$$

(ii) $$\begin{aligned} \forall x \in \mathcal{M} : \Psi(x, \mathbb{1}) &= x \\ \iff x\mathbb{1} &= x \\ \iff R_{\mathbb{1}} &= \mathrm{id}_{\mathcal{M}} \end{aligned}$$

using again $\Psi(x, g) \equiv xg$ *and the right translation* $R_g : \mathcal{M} \to \mathcal{M}$, $R_g(x) = xg$.

Remark. If $\Psi : \mathcal{M} \times G \to \mathcal{M}$ is an action from the right, than $\Phi(g, x) = \Psi(x, g^{-1}))$ is an action from the left, and vice versa if a left action Φ is given instead.

Definition 3.1.4. *Let* G *act on* $\mathcal{M}$ *from the left (or analogously from the right). Then the action is called free if*

$$\forall g \in G, \forall x \in \mathcal{M} : gx = x \implies g = \mathbb{1}. \tag{3.7}$$

Given a point $x \in \mathcal{M}$, *the set*

$$\begin{aligned} Gx &= \{y \in \mathcal{M} \mid \exists g \in G : y = gx\} \\ &\equiv \{gx \mid g \in G\} \end{aligned} \tag{3.8}$$

is called the orbit of the group action through the point x.

3.2 Lie algebras

Definition 3.2.1. *Let* $L_g : G \to G$ *be the left action of a Lie group upon itself (i.e.,* $L_g(g') = gg'$, *using the standard group multiplication). If* $\mathbb{1}$ *is the identity element of* G, *one can define the left-invariant vector field* $\bar{A} \in \mathfrak{X}(G)$ *determined by* $A \in T_{\mathbb{1}}G \equiv \mathfrak{g}$ *via*

$$\bar{A}_g = L_{g*}(A) \tag{3.9}$$

One can then define a commutator for $A, B \in \mathfrak{g}$, *using*

$$[A, B] = [\bar{A}, \bar{B}]_{\mathbb{1}} \in \mathfrak{g} \tag{3.10}$$

Note that this definition inherits the properties of the usual vector field commutator (2.8), that is, it is antisymmetric and obeys the jacobi identity. $(\mathfrak{g}, [\cdot, \cdot])$ *is then called the Lie algebra of* G.

Remark. More generally, a Lie algebra could be defined as any vector space carrying a Lie bracket $[\cdot, \cdot]$ that is bilinear, antisymmetric and obeys the Jacobi identity, without the notion of a Lie group. And although every Lie group canonically determines a Lie algebra, the converse is only true locally. This means that different Lie groups G and G' might have the same Lie algebra; the condition for this to be possible is that they are at least locally isomorphic and that they have the same universal covering group.

Definition 3.2.2. *The exponential map* $\exp : \mathfrak{g} \to G$ *is defined by*

$$\exp(A) \;=\; \gamma(1). \tag{3.11}$$

Here, γ is the map defined by $\gamma(t) = \varphi_t(\mathbb{1})$, where $\{\varphi_t\}$ is the one parameter group generated by $\bar{A}$. Using this definition, it can be shown that the exponential map satisfies $\gamma(t) = \exp(tA)$ and $\varphi_t(g) = g\gamma(t) = g\exp(tA)$.

Remark. There are some mathematical details necessary to show that exp is well defined. For example, one would have to show that $\bar{A}$ is a complete vector field and that the assignement $A \leftrightarrow \gamma$ is unique. For more information on this, see [16].

Example 3.2.3. If G is a matrix group as in 3.1.2, one can use the more familiar definition for the exponential map,

$$\exp : \mathrm{Mat}(n \times n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K}),\ \exp(A) \;=\; \sum_{k\,=\,0}^{\infty} \frac{A^k}{k!}, \tag{3.12}$$

or analogously for subgroups of $\mathrm{GL}(n, \mathbb{K})$. It satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA) \;=\; A\exp(tA) \;=\; \exp(tA)A. \tag{3.13}$$

The Lie algebra $\mathfrak{g}$ can then be defined by

$$\mathfrak{g} \;=\; \{A \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid \exp(tA) \in G\ \forall t \in \mathbb{R}\} \tag{3.14}$$

This definiton allows us to calculate the Lie algebras for the matrix groups given in 3.1.2:

$$\mathfrak{gl}(n, \mathbb{K}) \;= \mathrm{Mat}(n \times n, \mathbb{K}) \tag{3.15}$$
$$\mathfrak{sl}(n, \mathbb{K}) \;= \{A \in \mathrm{Mat}(n \times n, \mathbb{K}) \mid \mathrm{tr}A \;=\; 0\} \tag{3.16}$$
$$\mathfrak{so}(n) \;= \left\{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A + A^T \;=\; 0\right\} \tag{3.17}$$
$$\mathfrak{u}(n) \;= \left\{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A + A^\dagger \;=\; 0\right\} \tag{3.18}$$
$$\mathfrak{su}(n) \;= \{A \in \mathfrak{u}(n) \mid \mathrm{tr}A \;=\; 0\} \tag{3.19}$$
$$\mathfrak{sp}(n) \;= \left\{A \in \mathrm{Mat}(n \times n, \mathbb{H}) \mid A + A^\dagger \;=\; 0\right\} \tag{3.20}$$

Note that we interpret these sets as real vector spaces, equipped with a Lie bracket $[\cdot,\cdot]$, in the case of matrix groups usually $[A, B] = AB - BA$. If we then choose a basis $\{E_a\}$ for one of these spaces, it will in general satisfy some commutation relation

$$[E_a, E_b] \;=\; f_{ab}^c E_c, \tag{3.21}$$

where the f_{ab}^c are called the structure constants.

For more information on the Lie algebra $\mathfrak{sp}(n)$, see appendix A.

3.3 Representations and Lie-algebra valued forms

Definition 3.3.1. *Let V be some vector space. We then call a map $\rho : G \to GL(V)$ a representation of the group G if $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$ (i.e., ρ is a group homomorphism). Note that a representation always implies a left action on the vector space via*

$$gv = \rho(g)v, \tag{3.22}$$

where $\rho(g) \in GL(V)$ acts on $v \in V$ in the usual way.

Example 3.3.2. An important example for a representation is the adjoint representation of a Lie group on its Lie algebra, which can be constructed as follows: For any $g \in G$, we can define the adjoint isomorphism $\mathrm{Ad}_g : G \to G$ via $\mathrm{Ad}_g(g') = gg'g^{-1}$. We can than look at the push forward of this map, i.e.,

$$(\mathrm{Ad}_g)_{*\mathbb{1}} \equiv \mathfrak{Ad}_g : \mathfrak{g} \to \mathfrak{g}. \tag{3.23}$$

The map $\mathfrak{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, $g \mapsto \mathfrak{Ad}_g$ is then called the adjoint representation of the group G. We can differentiate this once more, resulting in

$$\mathfrak{Ad}_{*\mathbb{1}} \equiv \mathfrak{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}). \tag{3.24}$$

This is then called the adjoint representation of the Lie algebra. It satisfies

$$\mathfrak{ad}(A)(B) = \frac{\partial^2}{\partial s \partial t} (\exp(tA)\exp(sB)\exp(-tA))\bigg|_{s,t\,=\,0} = [A, B], \tag{3.25}$$

given $A, B \in \mathfrak{g}$. We can interpret $\mathfrak{ad}(A)$ as a matrix acting on the vector space $\mathfrak{g}$. Its components with respect to some basis E_a are given by the structure constants:

$$\mathfrak{ad}(E_i)(E_j) = [E_i, E_j] = f^k_{ij} E_k \implies (\mathfrak{ad}(E_i))^k{}_j = f^k_{ij}. \tag{3.26}$$

Additional details on the construction of these representations can be found in [16].

Definition 3.3.3. *Given a Lie group G and its Lie algebra $\mathfrak{g}$, we can use the adjoint representation of the Lie algebra to define a symmetric bilinear form*

$$K(A, B) = \mathrm{tr}(\mathfrak{ad}(A)\mathfrak{ad}(B)) \implies K_{ij} = f^k_{il} f^l_{jk}, \tag{3.27}$$

called the Killing form. If this form is non-degenerate, the Lie group as well as its Lie algebra are called semisimple.

One can show that if the group is compact, the Killing form on $\mathfrak{g}$ is negative definite, and its negative defines a left invariant riemannian metric on the group, called the Cartan-Killing metric (i.e., $g_{ij} = f^k_{il} f^l_{kj}$).

Remark. If we use the Cartan-Killing metric to pull down the structure constants upper index (i.e., $f_{ijk} := g_{kl} f^l_{ij} = f^n_{km} f^m_{nl} f^l_{ij}$), the Jacobi identity (2.16) implies that f_{ijk} is totally antisymmetric (using $f_{ijk} = -f_{jik}$).

Definition 3.3.4. *If $\eta \in \Omega^k(\mathcal{M}, \mathfrak{g})$ and $\omega \in \Omega^l(\mathcal{M}, \mathfrak{g})$ are Lie algebra valued forms, we can define their commutator as*

$$[\eta, \omega](X_1, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_\sigma (-1)^\sigma [\eta(X_{\sigma(1)}, \dots, X_{\sigma(k)}), \omega(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})], \tag{3.28}$$

where σ are again the possible commutations of the indices. We can always write Lie algebra valued forms as $\eta = \tilde{\eta} \otimes A$, $\omega = \tilde{\omega} \otimes B$, using $\mathbb{R}$-valued forms $\tilde{\eta}$, $\tilde{\omega}$. The commutator is then

$$[\eta, \omega] = (\tilde{\eta} \wedge \tilde{\omega}) \otimes [A, B]. \tag{3.29}$$

If we now choose a basis for our Lie algebra, $\{E_i\}$, this can be further decomposed into $\eta = \eta^i \otimes E_i$, $\omega = \omega^i \otimes E_i$, now with dim$(\mathfrak{g})$ $\mathbb{R}$-valued forms $\{\eta^i\}$, $\{\omega^i\}$. This leads to

$$[\eta, \omega] = (\eta^i \wedge \omega^j) \otimes [E_i, E_j] = f^k_{ij} (\eta^i \wedge \omega^j) \otimes E_k. \tag{3.30}$$

Lemma 3.3.5. *The commutator of Lie algebra valued forms $\eta \in \Omega^i(\mathcal{M}, \mathfrak{g})$, $\omega \in \Omega^j(\mathcal{M}, \mathfrak{g})$ and $\rho \in \Omega^k(\mathcal{M}, \mathfrak{g})$, has similiar properties to the usual commutator:*

(i) $[\eta, \omega] = -(-1)^{ij} [\omega, \eta]$ *(symmetry)*

(ii) $(-1)^{ik} [[\eta, \omega], \rho] + (-1)^{kj} [[\rho, \eta], \omega] + (-1)^{ij} [[\omega, \rho], \eta] = 0$ *(Jacobi identity)*

This means that the $\mathfrak{g}$-valued forms are a graded Lie algebra. The exterior derivative satisfies a Leibniz rule:

$$d([\eta, \omega]) = [d\eta, \omega] + (-1)^i [\eta, d\omega] \tag{3.31}$$

Definition 3.3.6. *Let ρ be a representation of the Group G on some vector space V and let η be a k-form taking values in the same vector space. We can then define the wedge product for a Lie algebra valued l-form ω and η by*

$$\rho_*(\omega) \wedge \eta(X_1, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_\sigma (-1)^\sigma \rho_*(\omega(X_{\sigma(1)}, \dots, X_{\sigma(l)})) \cdot \eta(X_{\sigma(l+1)}, \dots, X_{\sigma(l+k)}), \tag{3.32}$$

where " · " means

$$\rho_*(A) \cdot v = \frac{\partial}{\partial t} \rho(\exp(At)) v \Big|_{t=0} \tag{3.33}$$

for $A \in \mathfrak{g}, v \in V$.

If the representation used is obvious from the context, we will omit the ρ_* *and simply write* $\omega \wedge \eta$ *for (3.32)*

Remark. If ρ is the adjoint representation we simply get $\rho_*(\omega) \wedge \eta = [\omega, \eta]$ for Lie algebra valued forms ω, η (i.e., the usual commutator 3.3.4).

4 Theory of principal fibre bundles

In the last chapter, we hinted at the possibility to generalize the notion of a connection to a one-form taking values in some Lie algebra that has a certain behaviour under coordinate transformations. This object will turn out to be equivalent to the gauge potential appearing in field theories, for example Maxwell's electrodynamics with the potential $\mathcal{A}_\mu$. In this case, the theory is invariant under U(1)-gauge transformations; the Lie algebra of U(1) is just the set of imaginary numbers (i.e., $\mathfrak{u}(1) = \left\{c \in \mathbb{C} \mid c = -c^\dagger\right\} = \{c \in \mathbb{C} \mid \mathrm{Re}(c) = 0\}$), and the gauge potential is a one-form taking values in this Lie algebra. However, the object thats of physical interest is not the potential, but the field strength $\mathcal{F}$; this will turn out to be the curvature of the connection one-form $\mathcal{A}$.

Instead of studying objects on some base manifold $\mathcal{M}$ that are invariant under transformations by a group G, one can derive the local expressions for $\mathcal{A}$ and $\mathcal{F}$ from more general objects living on another manifold $\mathcal{P}$ that – at least locally – looks like $\mathcal{M} \times G$. This construction allows a more geometric interpretation of gauge theories.

This chapter is mainly based on [16, 20, 21, 23, 24].

4.1 Principal fibre bundles

Definition 4.1.1. *Let G be a Lie group. A tripel $(\mathcal{P}, \pi, \mathcal{M})$ consisting of manifolds $\mathcal{M}, \mathcal{P}$ and a map $\pi : \mathcal{P} \to \mathcal{M}$ is called a principal fibre bundle with structure group G, or a principal G-bundle, if the following holds:*

- *G acts freely on $\mathcal{P}$ to the right*

- *The fibres* $\mathcal{P}_x = \pi^{-1}(x)$ *are just the orbits of the* G*-action, i.e.,*

$$\pi^{-1}(\pi(p)) \;=\; \{pg : g \in G\} \tag{4.1}$$

- *for every* $x \in \mathcal{M}$ *there exists an open neighborhood* $\mathcal{U} \subseteq \mathcal{M}$ *and a diffeomorphism*

$$\Phi \;=\; (\Phi^1, \Phi^2) : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G \tag{4.2}$$

such that $\Phi^1(p) = \pi(p)$ *and* $\Phi^2(pg) = \Phi^2(p)g$ *for all* $p \in \pi^{-1}(\mathcal{U})$, $g \in G$. Φ *is called a local trivialization.*

Remark. The fibres $\mathcal{P}_x$ are diffeomorphic to the Lie group G, but they do not carry a canonical group structure since the identification depends on the base point $p \in \mathcal{P}_x$, hence there is no unique identity element.

Definition 4.1.2. *Let* $(\mathcal{P}, \pi, \mathcal{M})$ *be a principal bundle and let* Φ, Ψ *be two local trivializations for the open neighborhoods* $\mathcal{U} \subseteq \mathcal{M}$ *and* $\mathcal{V} \subseteq \mathcal{M}$*, respectively. We than can define the transition function* $g_{uv} : \mathcal{U} \cap \mathcal{V} \to G$ *from* Φ *to* Ψ *as*

$$g_{uv}(x) \;=\; \Phi^2(p)\Psi^2(p)^{-1}, \tag{4.3}$$

where $x = \pi(p)$*. It satisfies*

(i) $g_{uu}(y) = \mathbb{1} \; \forall y \in \mathcal{U}$

(ii) $g_{vu}(y) = g_{uv}^{-1} \; \forall y \in \mathcal{U} \cap \mathcal{V}$

(iii) $g_{uv}(y)g_{vw}(y)g_{wu}(y) = \mathbb{1} \; \forall y \in \mathcal{U} \cap \mathcal{V} \cap \mathcal{W}$

Remark. g_{uv} is indeed well defined. If we would have taken a different point p in (4.3), say $p' = pg$ for some suitable group element g, we would get

$$g_{uv}(x) \;=\; \Phi^2(pg)\Psi^2(pg)^{-1} \;=\; \Phi^2(p)gg^{-1}\Psi^2(p)^{-1} \;=\; \Phi^2(p)\Psi^2(p)^{-1} \tag{4.4}$$

hence the definition does not depend on the choice of p.

Definition 4.1.3. *Similar to the definition 2.1.4 for sections of a vector bundle we can define a section of a principal fibre bundle* $(\mathcal{P}, \pi, \mathcal{M})$ *as a map* $\sigma : \mathcal{U} \subseteq \mathcal{M} \to \mathcal{P}$ *satisfying* $\pi(\sigma(x)) = x$ *for all* $x \in \mathcal{U}$*. We again say that* σ *is global if* $\mathcal{U} = \mathcal{M}$ *and local otherwise.*

Remark. There is a natural correspondence between local sections and local trivializations; if $\sigma : \mathcal{U} \to \mathcal{P}$ is local section, then

$$\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G, \qquad \Phi(\sigma(x)g) = (x, g) \tag{4.5}$$

is a local trivialization. On the other hand, a given local trivialization $\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G$ automatically gives rise to a local section $\sigma : \mathcal{U} \to P$ via $\sigma(x) = \Phi^{-1}(x, \mathbb{1})$. If there exists a global section (or, equivalently, a global trivialization), we call the principal fibre bundle trivial. This is equivalent to the statement that the bundle is isomorphic to $(\mathcal{M} \times G, pr_1, \mathcal{M})$.

Example 4.1.4. Let $\mathcal{M}$ be a n-dimensional manifold. Consider

$$L_x\mathcal{M} := \{u \in \mathrm{Hom}(\mathbb{R}^n, T_x\mathcal{M}) \mid u \text{ is invertible}\} \tag{4.6}$$

and set $L(\mathcal{M}) = \bigcup_{x\in\mathcal{M}} L_x(\mathcal{M})$. Then $\mathrm{GL}(n,\mathbb{R})$ acts freely on $L(\mathcal{M})$ from the right:

$$uA := u \circ A, \qquad A \in \mathrm{GL}(n,\mathbb{R}). \tag{4.7}$$

Given a chart $\varphi : \mathcal{U} \to \Omega$ on $\mathcal{M}$, we can define a local section on $L(\mathcal{M})$,

$$\sigma(x) : T_{\varphi(x)}\mathbb{R}^n \equiv \mathbb{R}^n \to T_x\mathcal{M}, \qquad \sigma(x) \;=\; \varphi^{-1}_{*\varphi(x)} \tag{4.8}$$

and a projection

$$\pi : L(\mathcal{M}) \to \mathcal{M}, \qquad \pi^{-1}(x) \;=\; T_x\mathcal{M}. \tag{4.9}$$

This turns $(L(\mathcal{M}), \pi, \mathcal{M})$ into a principal $\mathrm{GL}(n,\mathbb{R})$-bundle, called the frame bundle of $\mathcal{M}$.

Definition 4.1.5. *Let $(\mathcal{P}, \pi, \mathcal{M})$ be a principal fibre bundle with structure group G, and let $\mathcal{N}$ be a manifold on which G acts from the left. We can then define a right action on $\mathcal{P} \times \mathcal{N}$ via $(p, z)g = (pg, g^{-1}z)$. We can then look at the space of equivalence classes with respect to this group action, i.e.,*

$$\begin{aligned}(\mathcal{P} \times \mathcal{N})/\sim \;&=\; \{[p, z] \mid (p, z) \in \mathcal{P} \times \mathcal{N}\}\\ &=:\; \mathcal{P} \times_G \mathcal{N}.\end{aligned} \tag{4.10}$$

This space naturally carries a projection $\hat{\pi} : \mathcal{P} \times_G \mathcal{N} \to \mathcal{M}$, $\hat{\pi}[p, z] = \pi(p)$. We can then define the to $(\mathcal{P}, \pi, \mathcal{M})$ associated bundle of type $\mathcal{N}$ as $(\mathcal{P} \times_G \mathcal{N}, \hat{\pi}, \mathcal{M})$. The local trivializations are given by the principal bundle: If $\Phi = (\pi, \Phi^2)$ is a trivialization of $(\mathcal{P}, \pi, \mathcal{M})$, than $\hat{\Phi} : \hat{\pi}^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{N}$, $\hat{\Phi}([p, z]) = (\pi(p), \Phi^2(p)z)$ defines a trivialization of the associated bundle.

In the special case that V is a vector space carrying a representation $\rho : G \to GL(V)$ (and by extension a left action defined by $gv = \rho(g)v$), we write $\mathcal{P} \times_\rho V$ for the associated vector bundle. The fibres inherit their vector space structure from V via

$$[p, v] + [p, w] \;=\; [p, v + w] \tag{4.11}$$

and

$$a[p, v] = [p, av] \tag{4.12}$$

for $a \in \mathbb{K}$.

Definition 4.1.6. *Let V be a vector space, ρ a representation of the group G on this space, and $(\mathcal{P}, \pi, \mathcal{M})$ a principal G-bundle. We then write $\Omega^k(P, \rho)$ for the space of V-valued k-forms satisfying*

(i) $R_g^*\eta \;=\; \rho(g^{-1})\eta \;\forall g \in \Gamma$

(ii) $\pi_*(X_i) = 0 \implies \eta(X_1, \ldots, X_i, \ldots, X_k) = 0 \;\forall X_i \in \mathfrak{X}(P)$.

If $k = 0$, this reduces to

$$f(pg) \;=\; \rho(g^{-1})f(p) \tag{4.13}$$

for $f \in C^\infty(P, V)$.

Lemma 4.1.7. *There exists a canonical isomorphism $\Omega^k(M, P \times_\rho V) \to \Omega^k(P, \rho)$. If $\theta \in \Omega^k(M, P \times_\rho V)$ we can define $\tilde{\theta} \in \Omega^k(P, \rho)$ by*

$$[p, \tilde{\theta}(X_1, \ldots, X_k)] \equiv \theta(\pi_*(X_1), \ldots, \pi_*(X_k)) \;\forall p \in P,\; X_i \in T_pP \tag{4.14}$$

Conversely, if $\eta \in \Omega^k(P, \rho)$ is given, we can define $\hat{\eta} \in \Omega^k(M, P \times_\rho V)$ by

$$\hat{\eta}_x(V_1, \ldots, V_k) \equiv [p, \eta_p(X_1, \ldots, X_k)] \;\forall V_i \in T_xM, \tag{4.15}$$

where $p \in P$ with $\pi(p) = x$ and $X_i \in T_pP$ with $\pi_*(X_i) = V_i$.

4.2 Connections on principal bundles

We now have a sufficient mathematical vocabulary to define a connection on a principal bundle, with arbitrary gauge group G. The previous definition of a linear connection will then turn out to be equivalent to the special case where the principal bundle in question is the frame bundle defined in example 4.1.4, with structure group $\mathrm{GL}(n, \mathbb{R})$.

Here, we will always assume that a principal G-bundle $(\mathcal{P}, \pi, \mathcal{M})$ is given, and that $\mathfrak{g}$ is the Lie algebra of G.

Definition 4.2.1. *Let $X \in \mathfrak{g}$. We can then define the fundamental vector field $\widetilde{X} \in \mathfrak{X}(\mathcal{P})$ by*

$$\widetilde{X}(p) \;=\; \frac{d}{dt} p \exp(tX)\bigg|_{t\,=\,0}. \tag{4.16}$$

The fundamental vector field confers a Lie algebra homomorphism $X \in \mathfrak{g} \mapsto \widetilde{X} \in \mathfrak{X}(P)$, which we will also call the infinitesimal generator of the G-action on $\mathcal{P}$, or simply the $\mathfrak{g}$-action.

The subspace spanned by the $\widetilde{X}$ *can also be identified with the vertical space, defined by*

$$V_p \ = \ \{v \in T_p\mathcal{P} \mid \pi_*(v) \ = \ 0\}\,, \tag{4.17}$$

i.e., the space of vectors that do not leave the current fibre.

Remark. While the vertical space is well defined and unique for any principal bundle, the complementary horizontal space is not. The definition will depend on the choice of the identity element for every fibre, which is not predetermined by the definition of the bundle.

To make this precise, one needs a way to identify the different fibres with the Lie algebra. This can be accomplished by introducing a one-form $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called a connection on the principal bundle. The horizontal space will then be defined as the subspace of $T_p\mathcal{P}$ that satisfies $\mathcal{A}(v) = 0$.

Definition 4.2.2. *Let* R_g *be the right action of the group* G *defined on* $\mathcal{P}$ *and* $\mathfrak{Ad}$ *be the adjoint representation as in 3.3.2.* $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$ *is then called a connection on* $\mathcal{P}$ *if*

(i) $R_g^*\mathcal{A} = \mathfrak{Ad}(g^{-1})\mathcal{A}$ *for all* $g \in G$*. If* G *is a matrix group, this means that*

$$\mathcal{A}(R_{g*}(w)) \ = \ \mathfrak{Ad}(g^{-1})(\mathcal{A}(w)) \ = \ g^{-1}\mathcal{A}(w)g \tag{4.18}$$

for all $g \in G$*,* $w \in T_p\mathcal{P}$*.*

(ii) $\mathcal{A}\left(\widetilde{X}(p)\right) = X$ *for all* $X \in \mathfrak{g}$ *and all* $p \in \mathcal{P}$*.*

We will denote the space of all connections on $\mathcal{P}$ *by* $\mathcal{C}(\mathcal{P})$*.*

Given a connection $\mathcal{A}$*, we will define the horizontal space as*

$$H_p^{\mathcal{A}} \ = \ \{v \in T_p\mathcal{P} \mid \mathcal{A}(v) \ = \ 0\}\,, \tag{4.19}$$

i.e., the space of vectors on $\mathcal{P}$ *that are not identified with an infinitesimal group action. It satisfies*

$$R_{g*}H_p^{\mathcal{A}} \ = \ H_{pg}^{\mathcal{A}}. \tag{4.20}$$

The horizontal and vertical space can be used to decompose the whole tangent space into

$$T_p\mathcal{P} \ = \ V_p \oplus H_p^{\mathcal{A}}. \tag{4.21}$$

Note that this construction is not unique and depends on the choice of $\mathcal{A}$*.*

Remark. This definition for the connection has the advantage of being rather elegant. But as mentioned above, it is not the way that gauge potentials are usually defined in physics. Our previous idea for a connection as a one-form on the basis manifold with a certain behaviour under coordinate transformation will emerge if we consider the pull back via a local section, as provided in the next lemma:

Lemma 4.2.3. *Let $\mathcal{A} \in \mathcal{C}(P)$, and let Φ, Ψ be two local trivializations around $\mathcal{U}$ and $\mathcal{V}$, respectively, and σ_u, σ_v be the corresponding local sections. We can then define the local connection, or gauge potential, as $\mathcal{A}_u = \sigma_u^* \mathcal{A}$. It satisfies the transformation formula*

$$\mathcal{A}_v(Y_x) = L^{-1}_{g_{uv}(x)*}(g_{uv*}(Y_x)) + \mathfrak{Ad}((g_{uv}(x))^{-1})(\mathcal{A}_u(Y_x)) \tag{4.22}$$

for all $Y_x \in T_x\mathcal{M}$ and $x \in \mathcal{U} \cap \mathcal{V}$. Here, L_g denotes the left action from G upon itself and g_{uv} is the transition function. If G is a matrix group, this formula simplifies significantly to

$$A_v = g_{uv}^{-1} A_u g_{uv} + g_{uv}^{-1} dg_{uv}. \tag{4.23}$$

Conversely, if we assign to each local trivialization Φ a one-form $\mathcal{A}_u$ in such a way that they all satisfy (4.22), we can piece them together to a globaly well defined unique connection $\mathcal{A} \in \mathcal{C}(\mathcal{P})$.

Remark. Notice the similarity between (4.23) and our original results for the endomorphism valued one-form associated to the linear connection, (2.61). In the special case where $\mathcal{P}$ is the frame bundle, the transition functions represent a different choice of basis for the tangent space, consistent with the situation in (2.61).

Definition 4.2.4. *Given a connection $\mathcal{A}$, we can define the horizontal projection of any vector $v \in T_p\mathcal{P}$ by*

$$v^H = v - \widetilde{\mathcal{A}(v)}(p), \tag{4.24}$$

i.e., $v^H = v$ if $v \in H_p^{\mathcal{A}}$ and $v^H = 0$ if $v \in V_p$.

Using this definition, we can define the exterior covariant derivate $D^{\mathcal{A}} : \Omega^k(\mathcal{P}, \mathfrak{g}) \to \Omega^{k+1}(\mathcal{P}, \mathfrak{g})$ as

$$D^{\mathcal{A}}\eta(X_0, \ldots, X_k) = d\eta(X_0^H, \ldots, X_k^H). \tag{4.25}$$

Remark. Although the definition of the exterior covariant derivate depends on $\mathcal{A}$ we will usually omit the superscript when the choice of $\mathcal{A}$ is clear from the context.

Lemma 4.2.5. *Given a connection $\mathcal{A} \in \mathcal{C}(\mathcal{P})$, the exterior covariant derivative satisfies*

(i) $D^{\mathcal{A}}\eta \in \Omega^{k+1}(P, \rho)$ *for all* $\eta \in \Omega^k(P, V)$ *with* $R_g^*\eta = \rho(g^{-1})\eta$

(ii) $D^{\mathcal{A}}\eta = d\eta + \rho_*(\mathcal{A}) \wedge \eta$ *for* $\eta \in \Omega^k(P, \rho)$

If additionally σ *is a local section of* $\mathcal{P}$ *and again* $\eta \in \Omega^k(P, \rho)$*, then*

$$\sigma^* D^{\mathcal{A}}\eta \;=\; d\sigma^*\eta + \rho_*(\sigma^*\mathcal{A}) \wedge \sigma^*\eta \tag{4.26}$$

Definition 4.2.6. *The curvature of a connection* $\mathcal{A}$ *is defined by*

$$\mathcal{F}^{\mathcal{A}} \;=\; D^{\mathcal{A}}\mathcal{A} \tag{4.27}$$

Due to Lemma 4.2.5, we have $\mathcal{F}^{\mathcal{A}} \in \Omega^2(P, \mathfrak{Ad})$.

Theorem 4.2.7. *The curvature form satisfies*

(i) $\mathcal{F}^{\mathcal{A}} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$ *(Cartan structure equation)*

(ii) $D^{\mathcal{A}}\mathcal{F}^{\mathcal{A}} = 0$ *(Bianchi identity)*

(iii) $D^{\mathcal{A}}D^{\mathcal{A}}\eta = \rho_*(\mathcal{F}^{\mathcal{A}}) \wedge \eta$

for a form $\eta \in \Omega^k(P, \rho)$.

Proof. These follow from direct computation. □

Remark. The equivalence of linear and gauge connections can now be made precise by utilizing the isomorphism $\xi \mapsto \tilde{\xi}$ in lemma 4.1.7. If $\mathcal{E} = \mathcal{P} \times_\rho V$ is the vector bundle associated to $(\mathcal{P}, \pi, \mathcal{M})$ with respect to some representation ρ on some vector space V, we can interpret the exterior covariant derivate on $\mathcal{P}$ as a map $\nabla^{\mathcal{A}} : \Gamma(\mathcal{E}) \to \Omega^1(\mathcal{M}, \mathcal{E})$ by setting

$$\nabla^{\mathcal{A}} \equiv D^{\mathcal{A}}, \tag{4.28}$$

in the sense that

$$\widetilde{\nabla^{\mathcal{A}}\xi} \;=\; D^{\mathcal{A}}\tilde{\xi},\ \forall \xi \in \Gamma(\mathcal{E}). \tag{4.29}$$

One can show that this indeed defines a linear connection on the vector bundle $\mathcal{E}$, as in 2.4.1. In fact, every linear connection can be interpreted as induced by a gauge connection (cf. [20,21] for further details). Given a local section $\sigma : \mathcal{U} \to \mathcal{P}$, the covariant derivate of an element $\xi \in \Gamma(\mathcal{E})$ along the vector $X \in \mathfrak{X}(\mathcal{M})$ can be calculated via

$$(\nabla^{\mathcal{A}}_X\xi)(x) \;=\; [\sigma(x), \mathrm{d}v_x(X_x) + \rho_*(\sigma^*\mathcal{A}(X_x))v(x)], \tag{4.30}$$

where $v \in C^\infty(\mathcal{U}, V)$ such that $\xi|_{\mathcal{U}} = [\sigma, v]$. This can be abbreviated by writing

$$\nabla^{\mathcal{A}}_X \xi \;=\; X(\xi) + \rho_*(\sigma^* \mathcal{A}(X))\xi, \tag{4.31}$$

which yields our original formula (2.57) if V is the tensor product space $T^p_q\mathcal{M}$, carrying the tensor representation ρ.

In the special case where $V = \mathcal{P} \times_{\mathfrak{Ad}} \mathfrak{g} := \mathfrak{Ad}\ \mathcal{P}$, we can identify the curvature form itself with a two form on the base manifold (i.e., $\mathcal{F} \in \Omega^2(\mathcal{M}, \mathfrak{Ad}\ \mathcal{P})$), on which we can act with $\nabla^{\mathcal{A}}$. In general, covariant derivatives for such forms $(X \otimes \eta) \in \Omega^k(\mathcal{M}, \mathfrak{Ad}\ \mathcal{P})$ with $X \in \Gamma(\mathfrak{Ad}\ \mathcal{P})$ can be computed by

$$D^{\mathcal{A}}(X \otimes \eta) \;=\; \nabla^{\mathcal{A}}(X) \wedge \eta + X \otimes \mathrm{d}\eta. \tag{4.32}$$

Lemma 4.2.8. *The relations in theorem 4.2.7 also hold if we write* $\mathcal{F}^{\mathcal{A}} \in \Omega^2(\mathcal{M}, \mathfrak{Ad}\ \mathcal{P})$*, meaning that*

(i) $\nabla^{\mathcal{A}}\mathcal{F}^{\mathcal{A}} = 0$

(ii) $\nabla^{\mathcal{A}}\nabla^{\mathcal{A}}\eta = \rho_*(\mathcal{F}^{\mathcal{A}}) \wedge \eta.$

Given a local section $\sigma : \mathcal{U} \to \mathcal{P}$*, the local field strength* $\mathcal{F}_u := s^*\mathcal{F}^{\mathcal{A}}$ *also satisfies*

$$\mathcal{F}_u = d\mathcal{A}_u + \tfrac{1}{2}[\mathcal{A}_u, \mathcal{A}_u] \tag{4.33}$$

with the local gauge potential $\mathcal{A}_u$*.*

5 Yang-Mills theory

We now have all the necessary ingredients to formulate Yang-Mills theory. Similar to the familiar case of electrodynamics, the equations of motion for the gauge field strength $\mathcal{F}$ follow from an action principle; the key difference is now that we are not constraint to an U(1)-gauge theory, but can consider fields taking values in more general Lie algebras.

This chapter is primarily based on the presentation in [20, 21]. The Yang-Mills equations with torsion have been derived and analyzed in numerous papers (e.g., [25]).

Let $\mathcal{M}$ be a closed, oriented, n-dimensional Riemannian manifold and G a compact and connected Lie group. Further, let $\langle\cdot,\cdot\rangle$ be an $\mathfrak{Ad}$-invariant scalar product on $\mathfrak{g}$, meaning that

$$\langle \mathfrak{Ad}(g)X_1, \mathfrak{Ad}(g)X_2\rangle \;=\; \langle X_1, X_2\rangle \qquad \forall g \in G, \forall X_1, X_2 \in \mathfrak{g}, \tag{5.1}$$

and write $\|X\|^2 \;=\; \langle X, X\rangle$. Our usual choice will be the trace for matrix groups (or its real part, respectively, for complex valued matrices). Suppose that the volume form of $\mathcal{M}$ is given by

$$\mathrm{d}\mathcal{M} = e^1 \wedge \cdots \wedge e^n \tag{5.2}$$

with respect to some positivly oriented orthonormal basis of the tangent space $\{E_a\}$ and its dual basis $\{e^a\}$.

Let $(\mathcal{P}, \pi, \mathcal{M})$ be a principle G-bundle over $\mathcal{M}$ and let $\mathfrak{Ad}\ \mathcal{P}$ be defined as above, such that the curvature of a given gauge connection can be interpreted as $\mathcal{F}^{\mathcal{A}} \in \Omega^2(\mathcal{M}, \mathfrak{Ad}\ \mathcal{P})$.

Definition 5.0.9. *The Yang-Mills action $S_{YM} : \mathcal{C}(\mathcal{P}) \to \mathbb{R}$ is defined as*

$$S_{YM}[\mathcal{A}] \;=\; \int_{\mathcal{M}} \|\mathcal{F}^{\mathcal{A}}\|^2 d\mathcal{M}, \tag{5.3}$$

where we extended the definition of the scalar product on $\mathfrak{g}$ to one for every fibre of $\mathfrak{Ad}\ \mathcal{P}$ by setting

$$\langle [p, X_1], [p, X_2] \rangle \;=\; \langle X_1, X_2 \rangle\,, \tag{5.4}$$

which is well defined due to the $\mathfrak{Ad}$-invariance of the original scalar product.

The action principle now tells us that physical fields are those for which the action becomes stationary, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{YM}[\mathcal{A} + t\eta]\Big|_{t=0} \;=\; 0 \qquad \forall \eta \in \Omega^1(\mathcal{M}, \mathfrak{Ad}\ \mathcal{P}). \tag{5.5}$$

This is equvialent to the requirement that $\mathcal{F}^{\mathcal{A}}$ satisfies a field equation:

Theorem 5.0.10. *$\mathcal{A}$ is a stationary point of S_{YM} if and only if*

$$D^{\mathcal{A}} * \mathcal{F} \;=\; 0. \tag{5.6}$$

Here, the Hodge star operator on $\mathfrak{Ad}\ \mathcal{P}$-valued forms can be constructed from the one on $\mathcal{M}$ by setting

$$*\,(X \otimes \eta) = X \otimes *_{\mathcal{M}}\eta \qquad \forall X \in \Gamma(\mathfrak{Ad}\ \mathcal{P}), \eta \in \Omega^k(\mathcal{M}), \tag{5.7}$$

and we will not distinguish between the two.

If we apply $*$ once more to the Yang-Mills equations (5.6), "$*\ D^{\mathcal{A}}\ *$" is just the covariant divergence with respect to $\mathcal{A}$, meaning that the Yang-Mills equations (5.6) are equivalent to

$$D_a \mathcal{F}^{ab} \;=\; E_a \mathcal{F}^{ab} + [\mathcal{A}_a, \mathcal{F}^{ab}] \;=\; 0. \tag{5.8}$$

In four dimensions, the Bianchi identity $D^{\mathcal{A}}\mathcal{F} = 0$ implies that the Yang-Mills equation is automatically satisfied if the field strength is self dual or anti-self dual:

$$*\,\mathcal{F} \;=\; \pm\mathcal{F} \implies D^{\mathcal{A}} * \mathcal{F} \;=\; \pm D^{\mathcal{A}}\mathcal{F} \;=\; 0. \tag{5.9}$$

This can be generalized to higher dimensions by introducing a 4-form $Q \in \Omega^4(\mathcal{M})$, such that

$$*\,\mathcal{F} \;=\; -*Q \wedge \mathcal{F}. \tag{5.10}$$

Taking the covariant derivative yields

$$D^{\mathcal{A}} * \mathcal{F} \;=\; -D^{\mathcal{A}}(*Q \wedge \mathcal{F}) \;=\; -(\underbrace{\mathrm{d} * Q}_{=:*\mathcal{H}} \wedge \mathcal{F} + *Q \wedge \underbrace{D^{\mathcal{A}}\mathcal{F}}_{=0}) \tag{5.11}$$

or equivalently

$$\mathrm{d} * \mathcal{F} + [\mathcal{A}, *\mathcal{F}] + *\mathcal{H} \wedge \mathcal{F} \;=\; 0. \tag{5.12}$$

In the case that $*\mathcal{H}\wedge\mathcal{F} = 0$, these are just the usual Yang-Mills equations. This can be achieved by demanding $*Q$ to be closed, which implies $*\mathcal{H} = 0$. If $*\mathcal{H} \wedge \mathcal{F} \neq 0$, we get an additional summand in our equations.

We will consider the case where the base manifold is equipped with a linear connection with totally antisymmetric torsion, and we can then identify the three-form $\mathcal{H}$ with the torsion-form of that connection; in this case, (5.12) are known as the torsion-full Yang-Mills equations. We can write it out in components by applying the Hodge star once more; $*(*\mathcal{H} \wedge \mathcal{F})$ then yields the contraction of $\mathcal{F}$ and $\mathcal{H}$.

Alternatively, we could start by considering the situation where we have a gauge connection $\mathcal{A}$ satisfying the usual Yang-Mills equation (5.6) and a torsion-full linear connection ∇; the covariant derivative with respect to this connection is then given by

$$\nabla_b \mathcal{F}^{bc} \;=\; E_b \mathcal{F}^{bc} + \omega^a_{ab}\mathcal{F}^{bc} + \omega^c_{ab}\mathcal{F}^{ab}, \tag{5.13}$$

where ω is the connection one-form corresponing to ∇. Then it makes sense to consider a covariant derivative $\tilde{\nabla}$ with respect to both connections, in the sense that

$$\tilde{\nabla}_b \mathcal{F}^{bc} \;=\; E_b \mathcal{F}^{bc} + \omega^a_{ab}\mathcal{F}^{bc} + \omega^c_{ab}\mathcal{F}^{ab} + [\mathcal{A}_b, \mathcal{F}^{bc}]. \tag{5.14}$$

Demanding this to vanish also yields the torsion-full Yang-Mills equations, now given in components by

$$\tilde{\nabla}_b \mathcal{F}^{bc} \;=\; E_b \mathcal{F}^{bc} + \omega^a_{ab}\mathcal{F}^{bc} + \omega^c_{ab}\mathcal{F}^{ab} + [\mathcal{A}_b, \mathcal{F}^{bc}] \;=\; 0. \tag{5.15}$$

These equations could also be derived directly from an action principle for the modified action

$$S \;=\; S_{YM} + S_{CS} \tag{5.16}$$

with the additional summand

$$S_{CS} \;=\; \int_{\mathcal{M}} \mathcal{F}^{\mathcal{A}} \wedge \mathcal{F}^{\mathcal{A}} \wedge *Q. \tag{5.17}$$

Further details can be found in [25, 26], where these actions appear in the context of string theory.

6

Yang-Mills equations on homogenous spaces

Having reiterated the mathematical foundations for Yang-Mills theory in the last chapters we will now begin to explore a special case where we hope to find explicit solutions for the torsion-full Yang-Mills equations. We will consider homogenous spaces of the form G/H with antisymmetric torsion and two distinct sets of generators, in a sense that will be explained below. Similar setups have been explored before (for example, see [10–15]); we will however make a slightly different ansatz, treating multiple cases simultanously.

6.1 Preliminaries

Let G be a compact semisimple Lie group and H a closed subgroup in such a way that $(G, \pi, G/H)$ is a reductive homogenous space. The Lie algebra of G can then be decomposed into

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \mathfrak{m} \tag{6.1}$$

where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is its complement. We will write $\{I_A\}$ for the set of all generators of $\mathfrak{g}$, i.e., we will use upper case letters from the beginning of the alphabet when we want to infer that a summation should include all generators. The subsets generating G/H and H will be denoted by $\{I_a\}$ and $\{I_i\}$, respectively, using lower case letters from the beginning or the middle of the alphabet. Since H is a closed subgroup of G, the commutation relations for the generators will then read

$$[I_i, I_j] \;=\; f^k_{ij} I_k, \qquad [I_i, I_a] \;=\; f^b_{ia} I_b, \qquad [I_a, I_b] \;=\; f^i_{ab} I_i + f^c_{ab}. \tag{6.2}$$

We will use the Cartan-Killing metric on $\mathfrak{g}$, and assume that our generators are choosen in such a way that

$$g_{AB} \;=\; f^C_{AD} f^D_{CB} \;=\; \delta_{AB}, \tag{6.3}$$

which means that

$$g_{ab} \;=\; \delta_{ab}, \qquad g_{ij} \;=\; \delta_{ij}, \qquad g_{ia} \;=\; 0. \tag{6.4}$$

This implies that both f_{ABC} and f_{AB}^{C} are totally antisymmetric[1].

Note that the metric on G also implies a metric on G/H, which can be constructed by considering the left invariant fields and forms provided by the generators. We write $\{\hat{E}_A\}$ for these vector fields on G and $\{\hat{e}^A\}$ for the corresponding dual one forms. Given some local section σ of the principal bundle $(G, \pi, G/H)$ we can pull these forms back to the basis G/H, and write

$$\sigma^* \hat{e}^A \equiv e^A \tag{6.5}$$

for the local one forms and E^A for the dual fields, respectively. Among these generators, the e^a span the space $T^*(G/H)$ and the remaining forms can be decomposed into $e^i = e^i_a e^a$.

These forms will obey the Maurer-Cartan equations 2.2.5, which for our choice of structure constants (6.2) read

$$\mathrm{d}e^a \;=\; -\, f_{ib}^{a} e^i \wedge e^b - \tfrac{1}{2} f_{bc}^{a} e^b \wedge e^c \tag{6.6}$$

$$\mathrm{d}e^i \;=\; -\, \tfrac{1}{2} f_{bc}^{i} e^b \wedge e^c - \tfrac{1}{2} f_{jk}^{i} e^j \wedge e^k \tag{6.7}$$

We can also use these local one forms to write the metric as

$$g_{G/H} \;=\; \delta_{ab} e^a \otimes e^b \tag{6.8}$$

We will consider a linear connection ω on the tangent bundle with non vanishing torsion proportional to the structure constants, i.e.,

$$T_{bc}^{a} \;=\; \varkappa f_{bc}^{a}. \tag{6.9}$$

Our choice of metric implies that the corresponding torsion tensor T_{abc} is totally antisymmetric (i.e., a three-form).
Employing the first structure equation (2.67) in conjunction with the Maurer-Cartan equations (6.6), this leads to

$$T^a \;=\; \mathrm{d}e^a + \omega^a{}_b \wedge e^b \tag{6.10}$$

$$\Longleftrightarrow \quad \tfrac{1}{2}\varkappa f_{cb}^{a} e^c \wedge e^b \;=\; -f_{ib}^{a} e^i \wedge e^b - \tfrac{1}{2} f_{cb}^{a} e^c \wedge e^b + \omega^a{}_b \wedge e^b \tag{6.11}$$

$$\Longleftrightarrow \quad \omega^a{}_b \;=\; f_{ib}^{a} e^i + \tfrac{1}{2}(\varkappa + 1) f_{cb}^{a} e^c, \tag{6.12}$$

or in other words

$$\omega_{cb}^{a} \;=\; e_c^i f_{ib}^{a} + \tfrac{1}{2}(\varkappa + 1) f_{cb}^{a}. \tag{6.13}$$

[1] One consequence of antisymmetry is that $f_{AB}^{A} = 0$ (no sum over a). For this it would suffice that the Cartan-Killing metric is diagonal, since we would have $f_{AB}^{A} = g^{AD} f_{ABD} = \alpha f_{ADA} = 0$, where α is the entry of g for the index A.

6.2 Yang-Mills equations on $\mathbb{R} \times G/H$

We will now consider the space $\mathbb{R} \times G/H$ equipped with a generalization of the linear connection (6.13) via

$$\omega^a_{cb} \;=\; e^i_c f^a_{ib} + \tfrac{1}{2}(\varkappa + 1) f^a_{cb} \quad \text{and} \quad \omega^0_{0b} \;=\; \omega^a_{0b} \;=\; \omega^0_{cb} \;=\; 0 \tag{6.14}$$

where we choose a coordinate τ on $\mathbb{R}$ and a basis one form $e^0 \;=\; \mathrm{d}\tau$. We further choose

$$g \;=\; e^0 \otimes e^0 + \delta_{ab} e^a \otimes e^b \tag{6.15}$$

as our metric on this space. We can use this metric to pull down the indices of the torsion and define the three form

$$\mathcal{H} \;=\; \frac{1}{3!} T_{abc} e^a \wedge e^b \wedge e^c, \qquad \textit{i.e.,} \qquad \mathcal{H}_{abc} \;=\; T_{abc} \;=\; \varkappa f_{abc}. \tag{6.16}$$

We are now interested in solutions of the torsion-full Yang-Mills equation (5.15) for a connection $\mathcal{A}$ on the trivial principal fibre bundle $((\mathbb{R} \times G/H) \times G, \pi, \mathbb{R} \times G/H)$ with structure group G (or equivalently, on the associated vector bundle $\mathcal{E}$). Using the local basis defined above, the connection and its field strength will be of the form

$$\mathcal{A} \;=\; \mathcal{A}_0 e^0 + \mathcal{A}_a e^a \quad \text{and} \quad \mathcal{F} \;=\; \mathcal{F}_{0a} e^0 \wedge e^a + \tfrac{1}{2} \mathcal{F}_{ab} e^a \wedge e^b. \tag{6.17}$$

We will choose the gauge in such a way that $\mathcal{A}_0 = 0$. In this situation, the torsion-full Yang-Mills equations are equivalent to

$$E_a \mathcal{F}^{a0} + \omega^a_{ab} \mathcal{F}^{b0} + [\mathcal{A}_a, \mathcal{F}^{a0}] \;=\; 0 \tag{6.18}$$

$$E_0 \mathcal{F}^{0c} + E_b \mathcal{F}^{bc} + \omega^a_{ab} \mathcal{F}^{bc} + \omega^c_{ab} \mathcal{F}^{ab} + [\mathcal{A}_a, \mathcal{F}^{ac}] \;=\; 0 \tag{6.19}$$

when written out in components.

To solve these equations we first need explicit terms for all the variables. For this, we restrict ourself to the case of G-invariant connections. The most general ansatz for a G-invariant connection[2] is given by

$$\mathcal{A} \;=\; e^i I_i + X_a e^a, \tag{6.20}$$

with the additional constraint that the X_a satisfy equation (B.1), which in this case implies

$$[I_i, X_a] \;=\; f^b_{ia} X_b. \tag{6.21}$$

[2] See appendix B for further information.

This condition is automatically satisfied if we makes an ansatz such that $X_a \sim I_a$, which we will do later on.

To calculate the field strength, we need expressions for $\mathrm{d}\mathcal{A}$ and $\mathcal{A} \wedge \mathcal{A}$. Differentiating (6.20) using the Maurer-Cartan equations leads to

$$\begin{aligned}
\mathrm{d}\mathcal{A} &= \mathrm{d}\left(e^i I_i + X_a e^a\right) \\
&= \mathrm{d}e^i I_i + \mathrm{d}X_a e^a + X_a \mathrm{d}e^a \\
&= -I_i \left(\tfrac{1}{2} f^i_{bc} e^b \wedge e^c + \tfrac{1}{2} f^i_{jk} e^j \wedge e^k\right) + \mathrm{d}X_a e^a \\
&\quad - X_a \left(f^a_{ib} e^i \wedge e^b + \tfrac{1}{2} f^a_{bc} e^b \wedge e^c\right) \\
&= -\tfrac{1}{2} I_i f^i_{bc} e^b \wedge e^c - \tfrac{1}{2} X_a f^a_{bc} e^b \wedge e^c - \tfrac{1}{2} I_i f^i_{jk} e^j \wedge e^k \\
&\quad - X_a f^a_{ib} e^i \wedge e^b + \dot{X}_a e^0 \wedge e^a,
\end{aligned} \tag{6.22}$$

where $\dot{X}_a = \dfrac{\partial}{\partial \tau} X_a$.

For the second term we get

$$\begin{aligned}
\mathcal{A} \wedge \mathcal{A} &= \tfrac{1}{2}[\mathcal{A}_a, \mathcal{A}_b] e^a \wedge e^b \\
&= \tfrac{1}{2}[e^i_a I_i + X_a, e^j_b I_j + X_b] e^a \wedge e^b \\
&= \tfrac{1}{2}\left(e^i_a e^j_b [I_i, I_j] + e^i_a [I_i, X_b] + e^j_b [X_a, I_j] + [X_a, X_b]\right) e^a \wedge e^b \\
&= \tfrac{1}{2} f^k_{ij} I_k e^i \wedge e^j + \tfrac{1}{2} f^c_{ib} X_c e^i \wedge e^b + \tfrac{1}{2} f^c_{aj} X_c e^a \wedge e^j + \tfrac{1}{2}[X_a, X_b] e^a \wedge e^b \\
&= \tfrac{1}{2} f^k_{ij} I_k e^i \wedge e^j + f^a_{ib} X_a e^i \wedge e^b + \tfrac{1}{2}[X_a, X_b] e^a \wedge e^b.
\end{aligned} \tag{6.23}$$

Adding these up yields

$$\mathcal{F} = \underbrace{\dot{X}_b}_{\mathcal{F}_{0b}} e^0 \wedge e^b - \tfrac{1}{2} \underbrace{\left(f^i_{bc} I_i + f^a_{bc} X_a - [X_b, X_c]\right)}_{\mathcal{F}_{bc}} e^b \wedge e^c \tag{6.24}$$

for the field strength. Using this, we get

$$E_a \mathcal{F}^{a0} = 0 \tag{6.25}$$

$$\omega^a_{ab} \mathcal{F}^{b0} = -e^i_a f^a_{ib} \dot{X}^b + \tfrac{1}{2}(\varkappa + 1) \underbrace{f^a_{ab}}_{=0} \dot{X}^b \tag{6.26}$$

$$[\mathcal{A}_a, \mathcal{F}^{a0}] = -e^i_a f^{ab}_i \dot{X}_b - [X_a, \dot{X}^a] \tag{6.27}$$

for the summands in (6.18). If we use the Cartan-Killing metric to pull down the indices in (6.26) and (6.27), the terms proportional to e^i_a cancel due to the total antisymmetry of the structure constants f_{ABC}[3], which means that the first set of Yang-Mills equations (6.18) is equivalent to

$$[X_a, \dot{X}^a] \;=\; 0. \tag{6.28}$$

This will also be solved by the same ansatz satisfying the G-invariance condition, $X_a \sim I_a$.

For the second set of equations (6.19), we need

$$\begin{aligned}
E_0\mathcal{F}^{0c} &= \ddot{X}^c && (6.29)\\
E_b\mathcal{F}^{bc} &= 0 && (6.30)\\
\omega^a_{ab}\mathcal{F}^{bc} &= \left(e^i_a f^a_{ib} + \tfrac{1}{2}(\varkappa+1)f^a_{ab}\right)\left(-f^{bck}I_k - f^{bce}X_e + [X^b, X^c]\right)\\
&= -\, e^i_a f^a_{ib} f^{bck} I_k - e^i_a f^a_{ib} f^{bce} X_e + e^i_a f^a_{ib}[X^b, X^c] && (6.31)\\
\omega^c_{ab}\mathcal{F}^{ab} &= \left(e^i_a f^c_{ib} + \tfrac{1}{2}(\varkappa+1)f^c_{ab}\right)\left(-f^{abk}I_k - f^{abe}X_e + [X^a, X^b]\right)\\
&= -\, e^i_a f^c_{ib} f^{abk} I_k - e^i_a f^c_{ib} f^{abe} X_e + e^i_a f^c_{ib}[X^a, X^b]\\
&\quad + \tfrac{1}{2}(\varkappa+1)\left(-f^c_{ab}f^{abk}I_k - f^c_{ab}f^{abe}X_e + f^c_{ab}[X^a, X^b]\right) && (6.32)\\
[\mathcal{A}_a, \mathcal{F}^{ac}] &= [e^i_a I_i + X_a, -f^{acj}I_j - f^{acb}X_b + [X^a, X^c]]\\
&= -\, e^i_a f^{acj}[I_i, I_j] - e^i_a f^{acb}[I_i, X_b] + e^i_a[I_i, [X^a, X^c]]\\
&\quad - f^{acj}[X_a, I_j] - f^{acb}[X_a, X_b] + [X_a, [X^a, X^c]]\\
&= -\, e^i_a f^{acj} f^k_{ij} I_k - e^i_a f^{acb} f^e_{ib} X_e + e^i_a\left([X^a, [X^c, I_i]] - [X^c, [I_i, X^a]]\right)\\
&\quad - f^{acj} f^b_{aj} X_b - f^{acb}[X_a, X_b] + [X_a, [X^a, X^c]]\\
&= -\, e^i_a f^{acj} f^k_{ij} I_k - e^i_a f^{acb} f^e_{ib} X_e + e^i_a f^{cb}_i [X^a, X_b] - e^i_a f^{ab}_i [X^c, X_b]\\
&\quad - f^{acj} f^b_{aj} X_b - f^{acb}[X_a, X_b] + [X_a, [X^a, X^c]] && (6.33)
\end{aligned}$$

Adding these up allows us to compute (6.19), which is now given by:

$$\begin{aligned}
\ddot{X}^c &= e^i_a\left(f^a_{ib}f^{bck} + f^c_{ib}f^{abk} + f^{acj}f^k_{ij}\right)I_k && (6.34)\\
&\quad + e^i_a\left(f^a_{ib}f^{bce} + f^c_{ib}f^{abe} + f^{acb}f^e_{ib}\right)X_e && (6.35)\\
&\quad - e^i_a\left(f^a_{ib}[X^b, X^c] + f^c_{ib}[X^a, X^b] + f^{cb}_i[X^a, X_b] + f^{ab}_i[X_b, X^c]\right) && (6.36)\\
&\quad - \tfrac{1}{2}(\varkappa+1)\left(-f^c_{ab}f^{abk}I_k - f^c_{ab}f^{abe}X_e + f^c_{ab}[X^a, X^b]\right) && (6.37)\\
&\quad + f^{acj}f^b_{aj}X_b + f^{acb}[X_a, X_b] - [X_a, [X^a, X^c]] && (6.38)
\end{aligned}$$

[3] If we instead consider arbitrary g_{ab} for our Cartan-Killing metric we still get $-e^i_a f^a_{ib}\dot{X}^b - e^i_a f^{ab}_i \dot{X}_b = e^i_a g^{ac} g^{bd} \dot{X}_d (f_{ibc} + f_{icb}) = 0$.

If we now pull down all the indices using our metric (6.15), we see that (6.34) and (6.35) vanish due to the Jacobi identity (2.16). (6.36) is zero as well, again using total antisymmetry of the structure constants.
Furthermore, the term $f^c_{ab}f^{abk}$ in (6.37) vanishes since we have $f^k_{aj} = 0$ and $f^C_{AD}f^D_{CB} = \delta_{AB}$ (i.e., $f^c_{ab}f^{abk} = \delta^k_c = 0$).

This means that the second set of Yang-Mills equations for this case simplify to

$$\begin{aligned}\ddot{X}_c \;=\; & \left(\tfrac{1}{2}(\varkappa+1)f_{abc}f_{abe} - f_{ajc}f_{aje}\right)X_e - \tfrac{1}{2}(\varkappa+3)f_{abc}[X_a,X_b] \\ & - [X_a,[X_a,X_c]]\end{aligned} \tag{6.39}$$

Remark. We could also use a more general ansatz for our metric. As long as the structure constants remain totally antisymmetric, the arguments for the vanishing of (6.34), (6.35) and (6.36) still hold, and the term $f^c_{ab}f^{abk}$ in (6.37) is zero as long as we use the Cartan-Killing metric and impose $g_{aj} = 0$ (i.e., we don't need $g_{ab} = \delta_{ab}$). The final equations then read

$$\begin{aligned}\ddot{X}^c \;=\; & \left(\tfrac{1}{2}(\varkappa+1)f^c_{ab}f^{abe} + f^{acj}f^e_{aj}\right)X_e - \tfrac{1}{2}(\varkappa+1)f^c_{ab}[X^a,X^b] \\ & + f^{acb}[X_a,X_b] - [X_a,[X^a,X^c]].\end{aligned} \tag{6.40}$$

The reason why this is of interest is that a choice of metric coefficients for the Cartan-Killing metric implies a choice of generators. It has been argued in [13, 14] that such a choice implies different geometric structures (e.g., if $G/H = \mathrm{SU}(3)/\mathrm{SU}(2) \equiv S^5$, our choice corresponds to an α-Sasakian manifold with $\alpha = -\frac{1}{2}$.). Starting with the assumption of the existence of a G-structure is a common approach (cf. [10–15]), and writing our equations in this way would make it easier to directly compare our results. That being said, we will continue to work with the flat metric to ease computations. Appendix C contains a short computation for S^5, showing that the final equations for different α-Sasakian structures on SU(3)/SU(2) are equivalent.

6.3 Splitting up the generators

We will now further specialize to the case where the generators of the coset G/H can be split into two distinct sets which will be denoted by $\{I_a\} = \{I_{a'}\} \cup \{I_{a''}\}$ (i.e., we will use single primes for one of the sets and double primes for the other). This ansatz encompasses many different types of manifolds. For example, if we choose $a' \in \{2,\dots,\dim(G/H)-1\}$, $a'' = 1$ we get an α-Sasakian manifold, whereas $a' \in \{4,\dots,\dim(G/H)-3\}$, $a'' = \{1,2,3\}$ would be suitable to describe a 3-Sasakian manifold. We will explore some examples in chapter 7.

In this case we can make the ansatz

$$\begin{aligned} \mathcal{A} &= e^i I_i + X_{a'} e^{a'} + X_{a''} e^{a''} \\ &= e^i I_i + \phi I_{a'} e^{a'} + \psi I_{a''} e^{a''}, \end{aligned} \tag{6.41}$$

i.e.,

$$X_{a'} \;=\; \phi I_{a'}, \qquad X_{a''} \;=\; \psi I_{a''} \tag{6.42}$$

where ϕ, ψ are functions of the real parameter τ. This ansatz automatically satisfies the G-invariance condition (6.21) as well as the first set of Yang-Mills equations (6.28), since both X_a and $\dot{X}_a$ are proportional to I_a.

We will further assume that the structure constants satisfy

$$f_{a'cd} f_{b'cd} \;=\; \alpha' \delta_{a'b'}, \qquad f_{a''cd} f_{b''cd} \;=\; \alpha'' \delta_{a''b''} \tag{6.43}$$

$$f_{a'ci} f_{b'ci} \;=\; \tfrac{1}{2}\left(1-\alpha'\right) \delta_{a'b'}, \qquad f_{a''ci} f_{b''ci} \;=\; \tfrac{1}{2}\left(1-\alpha''\right) \delta_{a''b''} \tag{6.44}$$

and that a similar condition holds true if the summation only runs over a subset:

$$\begin{aligned} f_{a'c'd'} f_{b'c'd'} &= \alpha'_1 \delta_{a'b'}, & f_{a''c''d''} f_{b''c''d''} &= \alpha''_1 \delta_{a''b''} \\ f_{a'c'd''} f_{b'c'd''} &= \alpha'_2 \delta_{a'b'}, & f_{a''c''d'} f_{b''c''d'} &= \alpha''_2 \delta_{a''b''} \\ f_{a'c''d''} f_{b'c''d''} &= \alpha'_3 \delta_{a'b'}, & f_{a''c'd'} f_{b''c'd'} &= \alpha''_3 \delta_{a''b''} \\ f_{a'c'j} f_{b'c'j} &= \alpha'_4 \delta_{a'b'}, & f_{a''c''j} f_{b''c''j} &= \alpha''_4 \delta_{a''b''} \\ f_{a'c''j} f_{b'c''j} &= \alpha'_5 \delta_{a'b'}, & f_{a''c'j} f_{b''c'j} &= \alpha''_5 \delta_{a''b''} \end{aligned} \tag{6.45}$$

Notice that there is some redundancy here since we have

$$\alpha' \;=\; \alpha'_1 + 2\alpha'_2 + \alpha'_3, \qquad \alpha'' \;=\; \alpha''_1 + 2\alpha''_2 + \alpha''_3 \tag{6.46}$$

$$\tfrac{1}{2}\left(1-\alpha'\right) \;=\; \alpha'_4 + \alpha'_5, \qquad \tfrac{1}{2}\left(1-\alpha''\right) \;=\; \alpha''_4 + \alpha''_5 \tag{6.47}$$

These conditions are satisfied for all manifolds we are interested in, for example the spheres $\mathrm{SU}(n+1)/\mathrm{SU}(n) = S^{2n+1}$ and $\mathrm{Sp}(n+1)/\mathrm{Sp}(n) = S^{4n+3}$.

Our Yang-Mills equations (6.39) can now be rewritten into equations for ϕ and ψ. For this, we need to split up the summations over a, b, c into summs over a' and a''. To do this, we consider the equations for $\ddot{X}_{c'}$ (i.e., for $\ddot{\phi}$). The one for $\ddot{\psi}$ will follow analogously.

We notice that we have $\ddot{X}_{c'} = \ddot{\phi} I_{c'}$, which means that we can have no terms proportional to $X_{e''}$ on the right hand side of (6.39), since $X_{e''}$ cannot be proportional to $I_{c'}$. This leads to

$$\begin{aligned}
\ddot{X}_{c'} &= \left(\tfrac{1}{2}(\varkappa+1)f_{abc'}f_{abe'} - f_{ajc'}f_{aje'}\right)X_{e'} \\
&\quad + \left(\tfrac{1}{2}(\varkappa+1)f_{abc'}f_{abe''} - f_{ajc'}f_{aje''}\right)X_{e''} \qquad \} = 0 \\
&\quad - \tfrac{1}{2}(\varkappa+3)\left(f_{abc'}[X_a,X_b]\right) - [X_a,[X_a,X_{c'}]] \\
&= \left(\tfrac{1}{2}(\varkappa+1)\alpha' - \tfrac{1}{2}\left(1-\alpha'\right)\right)X_{c'} \\
&\quad - \tfrac{1}{2}(\varkappa+3)\left(f_{abc'}[X_a,X_b]\right) - [X_a,[X_a,X_{c'}]] \\
&= \left(\tfrac{1}{2}\left(\varkappa+2\right)\alpha' - \tfrac{1}{2}\right)X_{c'} \\
&\quad - \tfrac{1}{2}(\varkappa+3)\left(f_{abc'}[X_a,X_b]\right) - [X_a,[X_a,X_{c'}]] \qquad (6.48)
\end{aligned}$$

We can calculate the commutators

$$[X_{a'},X_{b'}] = [\phi I_{a'},\phi I_{b'}] = \phi^2(f_{a'b'e'}I_{e'} + f_{a'b'e''}I_{e''} + f_{a'b'i}I_i) \qquad (6.49)$$

$$[X_{a'},X_{b''}] = [\phi I_{a'},\psi I_{b''}] = \phi\psi(f_{a'b''e'}I_{e'} + f_{a'b''e''}I_{e''} + f_{a'b''i}I_i) \qquad (6.50)$$

$$[X_{a''},X_{b''}] = [\psi I_{a''},\psi I_{b''}] = \psi^2(f_{a''b''e'}I_{e'} + f_{a''b''e''}I_{e''} + f_{a''b''i}I_i) \qquad (6.51)$$

which means that

$$\begin{aligned}
f_{abc'}[X_a,X_b] &= \phi^2 f_{a'b'c'}(f_{a'b'e'}I_{e'} + f_{a'b'e''}I_{e''} + f_{a'b'i}I_i) \\
&+ 2\phi\psi f_{a'b''c'}(f_{a'b''e'}I_{e'} + f_{a'b''e''}I_{e''} + f_{a'b''i}I_i) \\
&+ \psi^2 f_{a''b''c'}(f_{a''b''e'}I_{e'} + f_{a''b''e''}I_{e''} + f_{a''b''i}I_i) \\
&= \left(\phi^2\alpha'_1 + 2\phi\psi\alpha'_2 + \psi^2\alpha'_3\right)I_{c'} \qquad (6.52)
\end{aligned}$$

and

$$\begin{aligned}
[X_a,[X_a,X_{c'}]] &= [X_{a'},[X_{a'},X_{c'}]] + [X_{a''},[X_{a''},X_{c'}]] \\
&= [X_{a'},\phi^2(f_{a'c'e'}I_{e'} + f_{a'c'e''}I_{e''} + f_{a'c'i}I_i)] \\
&+ [X_{a''},\phi\psi(f_{a''c'e'}I_{e'} + f_{a''c'e''}I_{e''} + f_{a''c'i}I_i)] \\
&= \phi^3\left(f_{a'c'e'}f_{a'e'd'} + f_{a'c'e''}f_{a'e''d'} + f_{a'c'i}f_{a'id'}\right)I_{d'} \\
&+ \phi\psi^2\left(f_{a''c'e'}f_{a''e'd'} + f_{a''c'e''}f_{a''e''d'} + f_{a''c'i}f_{a''id'}\right)I_{d'} \\
&= -\phi^3\left(\alpha'_1+\alpha'_2+\alpha'_4\right)I_{c'} - \phi\psi^2\left(\alpha'_2+\alpha'_3+\alpha'_5\right)I_{c'} \qquad (6.53)
\end{aligned}$$

Putting (6.52) and (6.53) into (6.48) yields a differential equation for ϕ

$$\begin{aligned}
\ddot{\phi} &= \left(\tfrac{1}{2}\left(\varkappa+2\right)\alpha' - \tfrac{1}{2}\right)\phi - \tfrac{1}{2}(\varkappa+3)\left(\phi^2\alpha'_1 + 2\phi\psi\alpha'_2 + \psi^2\alpha'_3\right) \qquad (6.54) \\
&+ \phi^3\left(\alpha'_1+\alpha'_2+\alpha'_4\right) + \phi\psi^2\left(\alpha'_2+\alpha'_3+\alpha'_5\right)
\end{aligned}$$

and analogously for ψ

$$\begin{aligned}\ddot{\psi} \;=\; & \left(\tfrac{1}{2}\left(\varkappa+2\right)\alpha'' - \tfrac{1}{2}\right)\psi - \tfrac{1}{2}(\varkappa+3)\left(\psi^2\alpha_1'' + 2\phi\psi\alpha_2'' + \phi^2\alpha_3''\right) \\ & + \psi^3\left(\alpha_1'' + \alpha_2'' + \alpha_4''\right) + \psi\phi^2\left(\alpha_2'' + \alpha_3'' + \alpha_5''\right)\end{aligned} \tag{6.55}$$

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7 Examples: The spheres S^{2n+1} and S^{4n+3}

We now have sufficiently simplified equations at our disposal which means that we can start to look for solutions. We will construct new solutions on the spheres $\mathrm{SU}(n+1)/\mathrm{SU}(n) = S^{2n+1}$ and $\mathrm{Sp}(n+1)/\mathrm{Sp}(n) = S^{4n+3}$. We will first look at the lowest nontrivial dimension since this allows us to write down explicit terms for the generators and we will then consider a generalization to higher dimensions.

To find these new solutions, we will interpret our equations (6.54), (6.55) as classical equations of motion stemming from a potential V. To do this, we will introduce rescaled functions (Ψ, Φ) that satisfy

$$\phi(\tau) = \mu\ \Phi(\lambda\tau), \qquad \psi(\tau) = \nu\ \Psi(\lambda\tau), \tag{7.1}$$

with some constants $\mu, \nu, \lambda \in \mathbb{R}$. If we then fix either Φ or Ψ to lie on a symmetry axis of the potential $V(\Psi, \Phi)$, our equations simplify further to a form that is well studied in the literature [27–29]. See appendix D for some information on these equations and their solutions. Furthermore, we will construct numerical solutions for some interesting values of $\varkappa$, which can be found in appendix E.

7.1 Example: SU(3)/SU(2) ≡ S^5

The five-Sphere can be written as a coset space SU(3)/SU(2) and is a popular example for a homogenous space. We can choose the Gell-Man matrices as anti-hermitian basis for the 8-dimensional Lie algebra $\mathfrak{su}(3)$

$$I_1 = \frac{\mathrm{i}}{6}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, I_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, I_3 = \frac{\mathrm{i}}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$I_4 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I_5 = \frac{-\mathrm{i}}{2\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$I_6 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, I_7 = \frac{\mathrm{i}}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, I_8 = \frac{\mathrm{i}}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.2}$$

where we normalized them in such a way that the Cartan-Killing metric satisfies

$$g_{AB} = \delta_{AB}. \tag{7.3}$$

It is immediatly apparent that I_6, I_7, I_8 are just the generators of $\mathfrak{su}(2)$ embedded into three-dimensional space (i.e., they span a $\mathfrak{su}(2)$ subalgebra), and that I_1 plays a special role as well. We get

$$\begin{aligned} f_{678} &= -\frac{1}{\sqrt{3}} \\ f_{231} &= f_{451} = -\frac{1}{2} \\ f_{426} &= -f_{356} = -f_{257} = f_{238} = -f_{458} = \frac{1}{2\sqrt{3}} \end{aligned}$$

as the only non-zero structure consants. If we further identify $\{a'\} = \{2, 3, 4, 5\}$, $\{a''\} = \{1\}$ and $\{i\} = \{6, 7, 8\}$ we see that only

$$\alpha' = \tfrac{1}{2}, \qquad \alpha'_2 = \alpha'_5 = \tfrac{1}{4} \quad \text{and} \quad \alpha'' = \alpha''_3 = 1 \tag{7.4}$$

are non-zero. If we put these values into our equations (6.54) and (6.55), we get

$$\ddot{\phi} = \frac{1}{4}\varkappa\phi - \frac{\varkappa+3}{4}\phi\psi + \frac{1}{2}\phi^3 + \frac{1}{4}\phi\psi^2 \tag{7.5}$$

$$\ddot{\psi} = \frac{\varkappa+1}{2}\psi - \frac{\varkappa+3}{2}\phi^2 + \psi\phi^2. \tag{7.6}$$

This simplifies slightly by setting $(\psi, \phi) = (2\Psi, \Phi)$, such that

$$\ddot{\Phi} = \frac{1}{4}\varkappa\Phi - \frac{\varkappa+3}{2}\Phi\Psi + \frac{1}{2}\Phi^3 + \Phi\Psi^2 = -\frac{\partial V}{\partial \Phi} \tag{7.7}$$

$$\ddot{\Psi} = \frac{\varkappa+1}{2}\Psi - \frac{\varkappa+3}{4}\Phi^2 + \Psi\Phi^2 = -\frac{\partial V}{\partial \Psi}, \tag{7.8}$$

which also allows us to write down a potential

$$V = -\frac{\varkappa+1}{4}\Psi^2 - \frac{1}{8}\varkappa\Phi^2 - \frac{1}{8}\Phi^4 + \frac{\varkappa+3}{4}\Phi^2\Psi - \frac{1}{2}\Psi^2\Phi^2. \tag{7.9}$$

We are now interested in solutions that start and end at critical points of V, since those are the solutions that will have finite action.

7.1.1 Critical points

The critical points of the potential V are given by the solutions to the algebraic equations

$$0 = \frac{1}{4}\varkappa\Phi - \frac{\varkappa+3}{2}\Phi\Psi + \frac{1}{2}\Phi^3 + \Phi\Psi^2 \tag{7.10}$$

$$0 = \frac{\varkappa+1}{2}\Psi - \frac{\varkappa+3}{4}\Phi^2 + \Psi\Phi^2. \tag{7.11}$$

This is solved by the trivial solution $(\Phi, \Psi) = (0, 0)$ for all $\varkappa$. Notice that $\Phi = 0$ automatically implies $\Psi = 0$, which means that we won't find any critical points on one of the axis. If we now assume $\Phi \neq 0$, we see that (7.10) implies

$$\Phi^2 = -\frac{1}{2}\varkappa + (\varkappa+3)\Psi - 2\Psi^2, \tag{7.12}$$

which we can use to replace Φ^2 in (7.11):

$$0 = -2\Psi^3 + \frac{3}{2}(\varkappa+3)\Psi^2 + \left(\frac{1}{2} - \frac{1}{4}(\varkappa+3)^2\right)\Psi + \frac{1}{8}(\varkappa+3)\varkappa. \tag{7.13}$$

This equation is solved by $\Psi = \frac{1}{2}$ for all $\varkappa$, which means that $(\Phi, \Psi) = (1, \frac{1}{2})$ and $(\Phi, \Psi) = (-1, \frac{1}{2})$ are critical points for all $\varkappa$.

We can use this knowledge to factorize (7.13), which leads to an quadratic equation for Ψ:

$$\left(\Psi - \frac{1}{2}\right)\left(8\Psi^2 - 2(3\varkappa+7)\Psi + \varkappa(\varkappa+3)\right) = 0 \tag{7.14}$$

This equation might have no real solutions, depending on $\varkappa$. Note that (7.12) implies that every solution for Ψ yields two possible values for Φ, as long as the right hand side of (7.12) is positive. It is now easy to see that (7.14) has the general solutions

$$\Psi_\pm = \frac{7+3\varkappa}{8} \pm \frac{1}{8}\sqrt{(7+3\varkappa)^2 - 8(3+\varkappa)\varkappa} \tag{7.15}$$

which are only real for

$$\varkappa \leq -9 - 4\sqrt{2} \quad \text{or} \quad \varkappa \geq -9 + 4\sqrt{2} \tag{7.16}$$

Notice that $\Psi_\pm = \frac{1}{2}$ for $\varkappa = \mp\sqrt{5}$, meaning that the critical points for these values and the one that exists for all $\varkappa$ are degenerated.
Additionally, the solutions $\Psi_\pm$ only both yield a positive value for Φ^2 if $\varkappa \leq -1$, whereas Ψ_- also works for $\varkappa \geq 0$. This means we have the following number of critical points in relation to $\varkappa$:

$\varkappa \in$	number of critical points
$(-\infty, -9-4\sqrt{2})$	7
$(-9-4\sqrt{2}, -9+4\sqrt{2})$	3
$(-9+4\sqrt{2}, -1)$	7
$(-1, 0)$	3
$(0, \infty)$	5

We are now interested in values of $\varkappa$ for which different critical points are degenerated, i.e. $V(\Phi_1, \Psi_1) = V(\Phi_2, \Psi_2)$ to find solutions that connect these values, see figure 7.1. Of special interest are the values $\varkappa = -3$ and $\varkappa = -1$, since they lead to further simplification of the equations (7.7) and (7.8). As can be seen in 7.2, these values correspond to additional symmetries and we can make our equations even simpler by fixing Ψ to lie on one of those symmetry axis:

$$\ddot{\Phi} = \frac{1}{2}\left(\Phi^3 - \Phi\right), \qquad \text{for} \quad \varkappa = -1, \Psi = \frac{1}{2} \tag{7.17}$$

$$\ddot{\Phi} = \frac{1}{2}\left(\Phi^3 - \frac{3}{2}\Phi\right), \qquad \text{for} \quad \varkappa = -3, \Psi = 0 \tag{7.18}$$

Those are kink-equations (see appendix D for additional information), which are solved by

$$\Phi(\tau) = \tanh\left(\tfrac{1}{2}\tau\right), \qquad \text{for} \quad \varkappa = -1, \Psi = \frac{1}{2} \tag{7.19}$$

$$\Phi(\tau) = \sqrt{\tfrac{3}{2}}\tanh\left(\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\tau\right), \qquad \text{for} \quad \varkappa = -3, \Psi = 0 \tag{7.20}$$

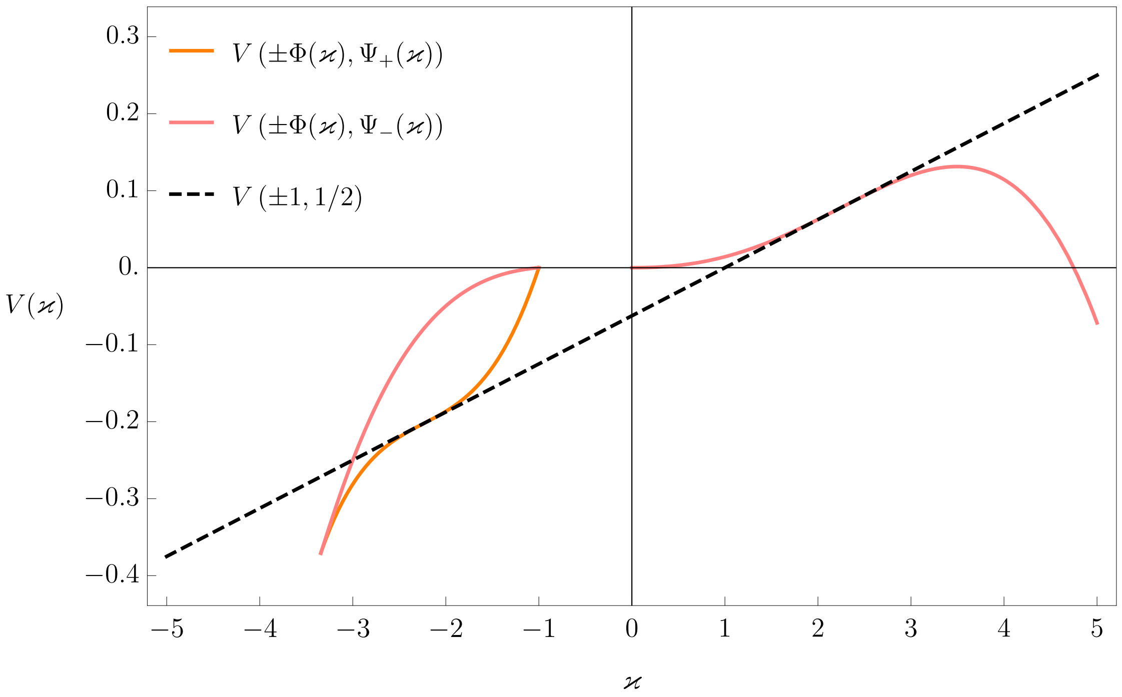
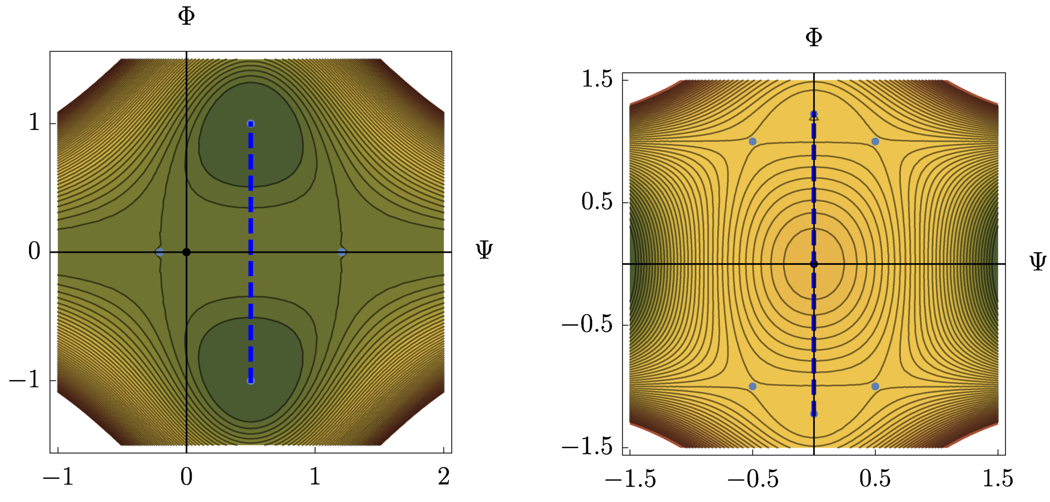


Figure 7.1: The potential V evaluated at all critical points as function of $\varkappa$. Notice that every line corresponds to two critical points and that there is another one for $V = \Phi = \Psi = 0$ for all $\varkappa$.



(a) $\varkappa = -1$ (b) $\varkappa = -3$

Figure 7.2: Contour plot of the potential V with analytical solutions for $\varkappa = -1, \varkappa = -3$.

As we can see in figure 7.1, $\varkappa = 0$, $\varkappa = 1$ and $\varkappa \approx 4.75$ are all possible candidates for solutions since the potential for some of the critical points are degenerated with the origin. However, there are no additional symmetries arising for these values, and we were not able to find analytical solutions. One can still construct numerical solutions, see e.g. figure E.1.

Note that although $\Phi = 0$ solves (7.7) and greatly simplifies (7.8), the resulting solutions for Ψ

$$\Psi(\tau) = \exp\left(\sqrt{\tfrac{\varkappa+1}{2}}\tau\right), \qquad \text{for} \quad \varkappa \geq -1 \tag{7.21}$$

$$\Psi(\tau) = \tau, \qquad \text{for} \quad \varkappa = -1 \tag{7.22}$$

$$\Psi(\tau) = \exp\left(\mathrm{i}\sqrt{\left|\tfrac{\varkappa+1}{2}\right|}\tau\right), \qquad \text{for} \quad \varkappa < -1 \tag{7.23}$$

do not have finite action and do not mediate between two critical points.

Remember that we rescaled Ψ to write down a potential; that means that we have to reverse this process to get the solutions for our original Yang-Mills equations.

7.1.2 Analytical solutions

In addition to the kink-solutions described above, we actually solved two other cases as well; If we had assumed that our base manifold is $S^1 \times G/H$ instead of $\mathbb{R} \times G/H$, we would search for periodic solutions to our equations. We could have also looked at the Lorentzian case $\mathrm{i}\mathbb{R} \times G/H, \tau \mapsto \mathrm{i}\tau$, which results in a sign flip in our equations; the solutions are then known as "bounces", and they have a physical interpretation as dyons. The procedure to find solutions for equations (7.17) and (7.18) for these cases are identical to the one described above (cf. appendix D). All solutions found for S^5 can be seen in table 7.1, where we already reversed the rescaling of Ψ; these are solutions to the original equations (7.5) and (7.6) (or their negatives, respectively).

Kink Solutions		
$\phi(\tau) = \tanh\left(\frac{1}{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{3}{2}}\tanh\left(\frac{1}{2}\sqrt{\frac{3}{2}}\tau\right)$	$\psi = 0$	$\varkappa = -3$
Bounce Solutions		
$\phi(\tau) = \sqrt{2}\ \operatorname{sech}\left(\sqrt{\frac{1}{2}}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{3}\ \operatorname{sech}\left(\frac{1}{2}\sqrt{3}\tau\right)$	$\psi = 0$	$\varkappa = -3$
Periodic Kink Solutions		
$\phi(\tau) = N(k)k\ \mathrm{sn}_k\left(\frac{1}{2}N(k)\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{3}{2}}N(k)k\ \mathrm{sn}_k\left(\frac{1}{2}\sqrt{\frac{3}{2}}N(k)\tau\right)$	$\psi = 0$	$\varkappa = -3$
Periodic Bounce Solutions		
$\phi(\tau) = \sqrt{2}\ M(k)k\ \mathrm{cn}_k\left(\sqrt{\frac{1}{2}}M(k)\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{3}\ M(k)k\ \mathrm{cn}_k\left(\frac{1}{2}\sqrt{3}M(k)\tau\right)$	$\psi = 0$	$\varkappa = -3$

Table 7.1: All solutions for $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$.

7.2 Example: $\mathrm{SU}(n+1)/\mathrm{SU}(n) \equiv S^{2n+1}$

The situation above can be generalized for spheres of arbitrary odd dimension $\mathrm{SU}(n+1)/\mathrm{SU}(n) \equiv S^{2n+1}$. We can construct a basis for $\mathrm{SU}(n+1)$ by embedding the basis for $\mathrm{SU}(n)$ into one higher dimension and adding matrices that only have values on the outermost rows and columns, plus one additional diagonal matrix. This is in fact how the Gell-Mann martices (7.2) are designed, where I_6, I_7, I_8 span the $\mathfrak{su}(2)$ subalgebra.

Remember that $\mathrm{SU}(n)$ is a (n^2-1) dimensional manifold. This means that the coset $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ will have $(2n+1)$ values for $\{a\}$ and the remaining (n^2-1) values for $\{i\}$. If we now name our indices

$$\{a'\} \;=\; \{2,\ldots,2n+1\}, \qquad \{a''\} \;=\; \{1\}, \qquad \{i\} \;=\; \{2n+2,\ldots,n^2-1\}, \tag{7.24}$$

the a' will correspond to the non-diagonal generators mentioned above and $a'' = 1$ will be the diagonal one. By construction of these generators, the structure constants satisfy

$$f_{a'b'c'} \;=\; f_{ia'1} \;=\; 0 \tag{7.25}$$

which in conjunction with the fact that there is only one index for a'' means that only α'_2, α'_4 and α''_3 are non-zero.

Using the definitions of a'_2 and a''_3 it is easy to see that the two are interconnected

$$a'_2\delta_{a'b'} \;=\; f_{a'c'd''}f_{b'c'd''} \implies 2n\;\alpha'_2 \;=\; f_{a'c'd''}f_{a'c'd''} \;=\; a''_3, \tag{7.26}$$

while (6.46) and (6.47) give us a relation between α'_2 and α'_4:

$$\alpha'_4 \;=\; \tfrac{1}{2}(1-2\alpha'_2) \tag{7.27}$$

Since we also have

$$g_{a''b''} \;=\; \delta_{a''b''} \;=\; f_{a''CD}f_{b''CD} \;=\; f_{a''c'd'}f_{b''c'd'} \;=\; \alpha''_3\delta_{a''b''} \tag{7.28}$$

we get

$$\alpha'_2 \;=\; \frac{1}{2n}, \qquad \alpha'_4 \;=\; \frac{n-1}{2n}, \qquad \alpha''_3 \;=\; 1 \tag{7.29}$$

This means that our Yang-Mills equations generalize to

$$\ddot{\phi} \;=\; \frac{1}{2}\left(\frac{\varkappa+2}{n}-1\right)\phi - \frac{\varkappa+3}{2n}\phi\psi + \frac{1}{2}\phi^3 + \frac{1}{2n}\phi\psi^2 \tag{7.30}$$

$$\ddot{\psi} \;=\; \frac{\varkappa+1}{2}\psi - \frac{\varkappa+3}{2}\phi^2 + \psi\phi^2 \tag{7.31}$$

We can again set $(\psi, \phi) = (\sqrt{2n}\Psi, \Phi)$, which yields

$$\ddot{\Phi} = \frac{1}{2}\left(\frac{\varkappa+2}{n} - 1\right)\Phi - \frac{\varkappa+3}{\sqrt{2n}}\Phi\Psi + \frac{1}{2}\Phi^3 + \Phi\Psi^2 = -\frac{\partial V}{\partial \Phi} \quad (7.32)$$

$$\ddot{\Psi} = \frac{\varkappa+1}{2}\Psi - \frac{\varkappa+3}{2\sqrt{2n}}\Phi^2 + \Psi\Phi^2 = -\frac{\partial V}{\partial \Psi} \quad (7.33)$$

and allows a potential

$$V = -\frac{1}{4}\left(\frac{\varkappa+2}{n} - 1\right)\Phi^2 + \frac{\varkappa+3}{2\sqrt{2n}}\Phi^2\Psi - \frac{1}{8}\Phi^4 - \frac{1}{2}\Phi^2\Psi^2 - \frac{\varkappa+1}{4}\Psi^2. \quad (7.34)$$

7.2.1 Critical points

The procedure to find solutions is completly analogoues to the special case discussed above, with the the difference that the numerical factors now depend on n. We get for example

$\varkappa \in$	number of critical points
$(-\infty, -1-4n-4\sqrt{n^2-n})$	7
$(-1-4n-4\sqrt{n^2-n}, -1-4n+4\sqrt{n^2-n})$	3
$(-1-4n+4\sqrt{n^2-n}, -1)$	7
$(-1, n-2)$	3
$(n-2, \infty)$	5

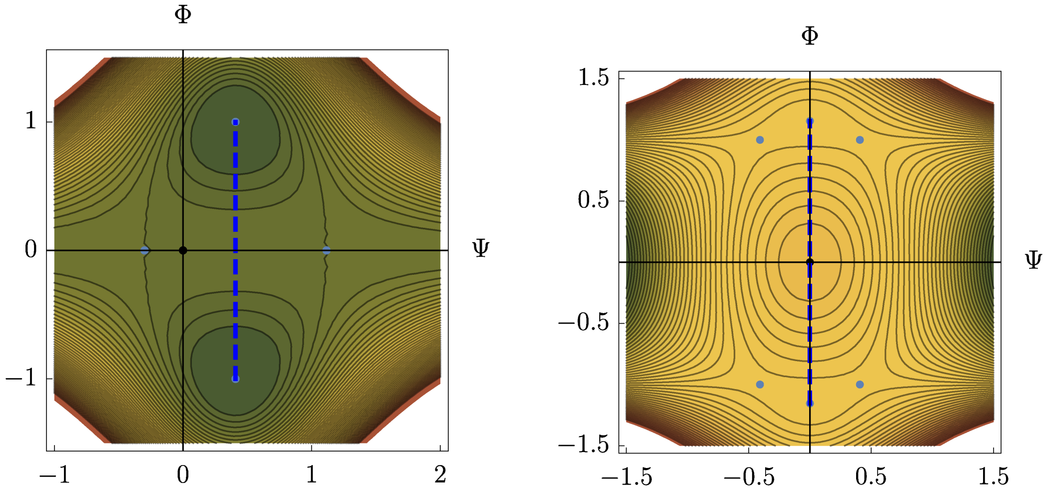
as new condition for the existence of critical points. $\varkappa = -3$ and $\varkappa = -1$ still yield the same symmetries (e.g., see figure 7.3 for $n = 3$), and we still find solutions for these values:

$$\Phi(\tau) = \tanh\left(\tfrac{1}{2}\tau\right), \qquad \text{for} \quad \varkappa = -1, \Psi = \frac{1}{\sqrt{2n}} \quad (7.35)$$

$$\Phi(\tau) = \sqrt{\tfrac{n+1}{n}}\tanh\left(\tfrac{1}{2}\sqrt{\tfrac{n+1}{n}}\tau\right), \qquad \text{for} \quad \varkappa = -3, \Psi = 0 \quad (7.36)$$

7.2.2 Analytical solutions

All analytical solutions for $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ can be found in table 7.2; this includes the dyonic case $\tau \to \mathrm{i}\tau$ and the periodic case $\mathbb{R} \to S^1$. These are again solutions to the original equations (7.30) and (7.31), without a rescaling.



(a) $\varkappa = -1$ (b) $\varkappa = -3$

Figure 7.3: Contour plot for V with analytical solutions for $\varkappa = -1$, $\varkappa = -3$ on $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ with $n = 3$, i.e. on S^7.

Kink Solutions		
$\phi(\tau) = \tanh\left(\frac{1}{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{n+1}{n}} \tanh\left(\frac{1}{2}\sqrt{\frac{n+1}{n}}\tau\right)$	$\psi = 0$	$\varkappa = -3$
Bounce Solutions		
$\phi(\tau) = \sqrt{2}\ \mathrm{sech}\left(\frac{1}{2}\sqrt{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{2(n+1)}{n}}\ \mathrm{sech}\left(\frac{1}{2}\sqrt{\frac{2(n+1)}{n}}\tau\right)$	$\psi = 0$	$\varkappa = -3$
Periodic Kink Solutions		
$\phi(\tau) = N(k)k\ \mathrm{sn}_k\left(\frac{1}{2}N(k)\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{n+1}{n}}N(k)k\ \mathrm{sn}_k\left(\frac{1}{2}\sqrt{\frac{n+1}{n}}N(k)\tau\right)$	$\psi = 0$	$\varkappa = -3$
Periodic Bounce Solutions		
$\phi(\tau) = \sqrt{2}\ M(k)k\ \mathrm{cn}_k\left(\frac{1}{2}\sqrt{2}M(k)\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{2(n+1)}{n}}M(k)k\ \mathrm{cn}_k\left(\frac{1}{2}\sqrt{\frac{2(n+1)}{n}}M(k)\tau\right)$	$\psi = 0$	$\varkappa = -3$

Table 7.2: All solutions for $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$.

7.3 Example: Sp(2)/Sp(1) $\equiv S^7$

Another coset space that is topologically S^7 is Sp(2)/Sp(1), where Sp(n) is the compact symplectic group. We will choose a basis for $\mathfrak{sp}(2)$ as dicussed in appendix A, normalized in such a way that $g_{AB} = \delta_{AB}$:

$$I_1 = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, I_2 = \frac{\mathrm{i}}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$I_3 = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, I_4 = \frac{\mathrm{i}}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$I_5 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, I_6 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$I_7 = \frac{\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, I_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I_9 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, I_{10} = \frac{\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (7.37)$$

If we now name our indices

$$\{a'\} = \{1,2,3,4\}, \qquad \{a''\} = \{5,6,7\} \quad \text{and} \quad \{i\} = \{8,9,10\} \qquad (7.38)$$

we see that

$$\alpha'_2 = \alpha'_4 = \frac{1}{4}, \qquad \alpha''_1 = \frac{2}{3}, \qquad \alpha''_3 = \frac{1}{3} \tag{7.39}$$

are the only non vanishing coefficients. This means our equations (6.54) and (6.55) now read

$$\ddot{\phi} = \frac{1}{12}\left(3\varkappa\phi - 3(\varkappa+3)\phi\psi + 6\phi^3 + 3\phi\psi^2\right) \tag{7.40}$$

$$\ddot{\psi} = \frac{1}{24}\left(12(\varkappa+1)\psi - 8(\varkappa+3)\psi^2 - 3(\varkappa+3)\phi^2 + 16\psi^3 + 6\psi\phi^2\right) \tag{7.41}$$

These can be simplified by setting $(\psi(\tau), \phi(\tau)) = \left(\Psi\left(\frac{1}{2\sqrt{3}}\tau\right), \frac{\sqrt{3}}{2}\Phi\left(\frac{1}{2\sqrt{3}}\tau\right)\right)$:

$$\ddot{\Phi} = 3\varkappa\Phi - 3(k+3)\Phi\Psi + \tfrac{9}{2}\Phi^3 + 3\Phi\Psi^2 \tag{7.42}$$

$$\ddot{\Psi} = 6(\varkappa+1)\Psi - 4(\varkappa+3)\Psi^2 - \frac{3(\varkappa+3)}{2}\Phi^2 + 8\Psi^3 + 3\Psi\Phi^2. \tag{7.43}$$

This also allows for a potential

$$V = -3(\varkappa+1)\Psi^2 + \frac{4(\varkappa+3)}{3}\Psi^3 - 2\Psi^4 - \frac{3}{2}\Psi^2\Phi^2 - \frac{3\varkappa}{2}\Phi^2 - \frac{9}{8}\Phi^4 + \frac{3(\varkappa+3)}{2}\Psi\Phi^2 \tag{7.44}$$

such that

$$-\frac{\partial V}{\partial \Phi} = \ddot{\Phi}, \qquad -\frac{\partial V}{\partial \Psi} = \ddot{\Psi}. \tag{7.45}$$

7.3.1 Critical points

First we observe that $(\Phi, \Psi) = (0, 0)$ still solves the algebraic equations for $\ddot{\Phi} = \ddot{\Psi} = 0$. However, $\Phi = 0$ no longer leads to $\Psi = 0$, but we are left with a quadratic expression for Ψ:

$$0 = 2\Psi\left(4\Psi^2 - 2(\varkappa+3)\Psi + 3(\varkappa+1)\right) \tag{7.46}$$

$$\Longleftrightarrow \quad \Psi = 0 \quad \text{or} \quad 0 = \Psi^2 - \frac{\varkappa+3}{2}\Psi + \frac{3(\varkappa+1)}{4} \tag{7.47}$$

$$\Longleftrightarrow \quad \Psi = 0 \quad \text{or} \quad \Psi_\pm = \frac{\varkappa+3}{4} \pm \frac{1}{4}\sqrt{\varkappa^2 - 6\varkappa - 3} \tag{7.48}$$

Notice that the potential is also symmetric under $\Phi \leftrightarrow -\Phi$, which means that if these critical points exist (i.e., $\varkappa^2 - 6\varkappa - 3 \geq 0$), they lie on a symmetry axis. Since there are no further critical points on this axis, the only possibile solutions for $\Phi = 0$ either mediate between Ψ_+ and Ψ_- or between $\Psi = 0$ and $\Psi_\pm$. The former is the case for $\varkappa = -3$, where Ψ_+ and Ψ_- become equidistant to the origin

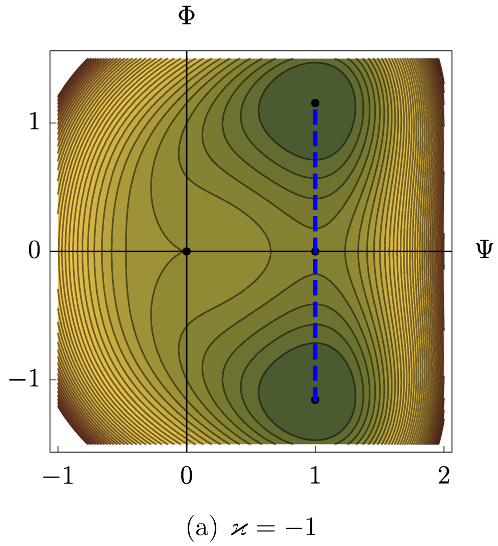


Figure 7.4: Contour plot of V for $\varkappa = -1$, $\varkappa = -3$ with two analytical solutions for $\varkappa = -3$ and one for $\varkappa = -1$ on $Sp(2)/Sp(1)$.

and we have an additional symmetry for the potential along the $\Psi = 0$ axis. Our equations for these cases simplify to

$$\ddot{\Phi} = \frac{9}{2}\left(\Phi^3 - 2\Phi\right), \qquad \text{for} \quad \varkappa = -3, \Psi = 0 \tag{7.49}$$

$$\ddot{\Psi} = 8\left(\Psi^3 - \frac{3}{2}\Psi\right), \qquad \text{for} \quad \varkappa = -3, \Phi = 0 \tag{7.50}$$

Those are both kink-equations, and we find the solutions

$$\Phi(\tau) = \sqrt{2}\tanh\left(\sqrt{\tfrac{9}{2}}\tau\right), \qquad \text{for} \quad \varkappa = -3, \Psi = 0 \tag{7.51}$$

$$\Psi(\tau) = \sqrt{\tfrac{3}{2}}\tanh\left(\sqrt{6}\tau\right), \qquad \text{for} \quad \varkappa = -3, \Phi = 0 \tag{7.52}$$

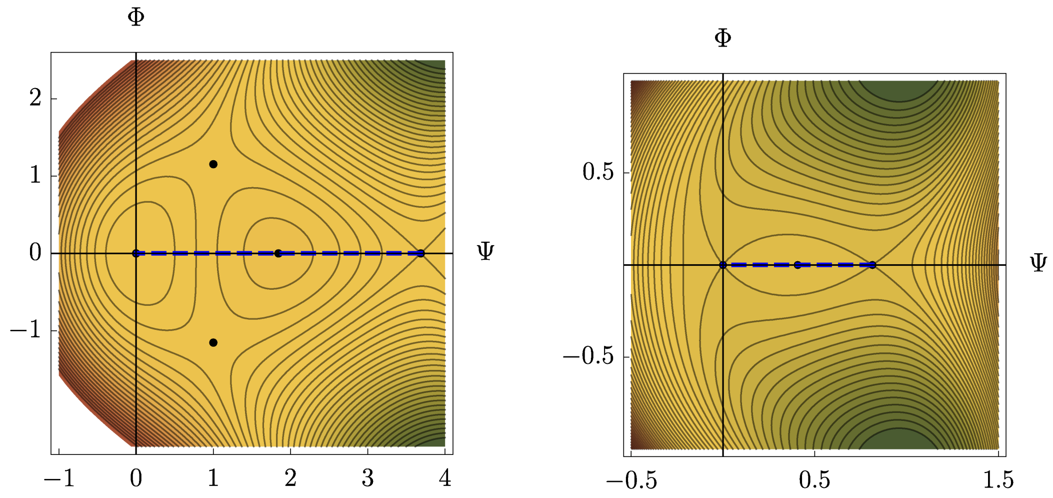
see also figure 7.4. $\varkappa = -3$ is in fact the only value for which $V(0, \Psi_-) = V(0, \Psi_+)$. We can also look for values of $\varkappa$ for which $V(0, \Psi_\pm) = V(0, 0)$, which leads to two solutions for $\varkappa$,

$$\varkappa_\pm = \frac{3}{4}\left(5 \pm \sqrt{33}\right), \tag{7.53}$$

for which $V(0, \Psi_+) = V(0, 0)$. Putting these values into (7.43) yields two radial kink equations

$$\ddot{\Psi} = \tfrac{1}{2}\Psi\left(16\Psi^2 - 6(9 + \sqrt{33})\Psi + 57 + 9\sqrt{33}\right) \tag{7.54}$$

$$\ddot{\Psi} = \tfrac{1}{2}\Psi\left(16\Psi^2 - 6(9 - \sqrt{33})\Psi + 57 - 9\sqrt{33}\right), \tag{7.55}$$



(a) $\varkappa_+ = \frac{3}{4}\left(5+\sqrt{33}\right)$ (b) $\varkappa_- = \frac{3}{4}\left(5-\sqrt{33}\right)$

Figure 7.5: Contour plot of V for $\varkappa_\pm$ with analytical solutions on $Sp(2)/Sp(1)$.

which are solved by

$$\Psi(\tau) = -\frac{6}{9-\sqrt{33}}\left(\tanh\left(\tfrac{1}{2}\sqrt{\tfrac{1}{2}(57+9\sqrt{33})}\tau\right)-1\right), \qquad \text{for} \quad \varkappa=\varkappa_+, \Phi=0 \tag{7.56}$$

$$\Psi(\tau) = -\frac{6}{9+\sqrt{33}}\left(\tanh\left(\tfrac{1}{2}\sqrt{\tfrac{1}{2}(57-9\sqrt{33})}\tau\right)-1\right), \qquad \text{for} \quad \varkappa=\varkappa_-, \Phi=0. \tag{7.57}$$

See figure 7.5.

If we now search for critical points with $\Phi \neq 0$, we get

$$\Phi^2 = -\frac{2}{3}\left(\Psi^2-(\varkappa+3)\Psi+\varkappa\right) \tag{7.58}$$

from the first equation (7.42), which we can put into (7.43) to get a third order equation for Ψ:

$$0 = 6\Psi^3-(\varkappa+3)\Psi^2-(\varkappa^2+2\varkappa+3)\Psi+\varkappa(\varkappa+3) \tag{7.59}$$

$$= (\Psi-1)(6\Psi^2-(k-3)\Psi-\varkappa(\varkappa+3)) \tag{7.60}$$

$$\Longleftrightarrow \quad \Psi = 1 \quad \text{or} \quad 0=6\Psi^2-(k-3)\Psi-\varkappa(\varkappa+3) \tag{7.61}$$

$$\Longleftrightarrow \quad \Psi = 1 \quad \text{or} \quad \tilde{\Psi}_\pm = \frac{1}{12}(\varkappa-3\pm\sqrt{25\varkappa^2+66\varkappa+9}) \tag{7.62}$$

Notice that $\Psi_+ = 1$ for $\varkappa = -2 \pm \sqrt{13}$.

Again, $\tilde{\Psi}_\pm$ only exists if the square root is real, which is the case for

$$\varkappa \notin (-\tfrac{3}{25}(11 + 4\sqrt{6}), -\tfrac{3}{25}(11 - 4\sqrt{6})) \approx (-2.5, 0.14) \tag{7.63}$$

and we have the additional condition that Φ^2 has to be positive, which further leads to the contraints

$$\varkappa \in (\approx -6.58, -\tfrac{1}{25}(33 + 12\sqrt{6})) \quad \text{or} \quad \varkappa >= 0, \qquad \text{for} \quad \tilde{\Psi}_+ \tag{7.64}$$

$$\varkappa \leq -\tfrac{1}{25}(33 + 12\sqrt{6}) \approx -2.5, \qquad \text{for} \quad \tilde{\Psi}_- \tag{7.65}$$

Similar to the case $SU(n)/SU(n-1)$, we again find critical points which are independent of $\varkappa$ at $(\Phi, \Psi) = (\pm\frac{2}{\sqrt{3}}, 1)$. For $\varkappa = -1$, both solutions $\tilde{\Psi}_\pm$ are imaginary, but we find that Ψ_+ from (7.48) takes the value $\Psi_+ = 1$ as well. This means that the two critcal points at $(\pm\frac{2}{\sqrt{3}}, 1)$ and the one at $(0, 1)$ lie on a straight line, and although the potential is no longer symmetric along this axis we still find a kink equation for these values:

$$\ddot{\Phi} \;=\; \tfrac{9}{2}\left(\Phi^3 - \tfrac{4}{3}\Phi\right), \quad \text{for} \quad \varkappa = -1, \Psi = 1, \tag{7.66}$$

which has the solution

$$\Phi(\tau) \;=\; \tfrac{2}{\sqrt{3}} \tanh\left(\sqrt{3}\tau\right), \quad \text{for} \quad \varkappa = -1, \Psi = 1, \tag{7.67}$$

as seen in figure 7.4.

7.3.2 Analytical solutions

We again list all analytical solutions to the original equations (7.40) and (7.41) in table 7.3. Notice that we still get the same solution for $\varkappa = -1$.

Kink Solutions		
$\phi(\tau) = \tanh\left(\frac{1}{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{3}{2}}\tanh\left(\frac{1}{2}\sqrt{\frac{3}{2}}\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{\frac{3}{2}}\tanh\left(\sqrt{\frac{1}{2}}\tau\right)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{6}{9\mp\sqrt{33}}\left(\tanh\left(\sqrt{\frac{1}{2}}w_\pm\tau\right) - 1\right)$	$\phi = 0$	$\varkappa = \frac{3}{4}\left(5 \pm \sqrt{33}\right)$
Bounce Solutions		
$\phi(\tau) = \sqrt{2}\,\mathrm{sech}\left(\frac{1}{2}\sqrt{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{3}\,\mathrm{sech}\left(\frac{1}{2}\sqrt{3}\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{3}\,\mathrm{sech}\,(\tau)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{6\sqrt{2}}{9\mp\sqrt{33}}\left(\mathrm{sech}\,(w_\pm\tau) - 1\right)$	$\phi = 0$	$\varkappa = \frac{3}{4}\left(5 \pm \sqrt{33}\right)$
Periodic Kink Solutions		
$\phi(\tau) = N(k)k\,\mathrm{sn}_k\left(\frac{1}{2}N(k)\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{3}{2}}N(k)k\,\mathrm{sn}_k\left(\frac{1}{2}\sqrt{\frac{3}{2}}N(k)\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{\frac{3}{2}}N(k)k\,\mathrm{sn}_k\left(\sqrt{\frac{1}{2}}N(k)\tau\right)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{6}{9\mp\sqrt{33}}\left(N(k)k\,\mathrm{sn}_k\left(\sqrt{\frac{1}{2}}w_\pm N(k)\tau\right) - 1\right)$	$\phi = 0$	$\varkappa = \frac{3}{4}\left(5 \pm \sqrt{33}\right)$
Periodic Bounce Solutions		
$\phi(\tau) = \sqrt{2}M(k)k\,\mathrm{cn}_k\left(\frac{1}{2}M(k)\sqrt{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{3}M(k)k\,\mathrm{cn}_k\left(\frac{1}{2}\sqrt{3}M(k)\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{3}M(k)k\,\mathrm{cn}_k\left(M(k)\tau\right)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{6\sqrt{2}}{9\mp\sqrt{33}}\left(M(k)k\,\mathrm{cn}_k\left(w_\pm M(k)\tau\right) - 1\right)$	$\phi = 0$	$\varkappa = \frac{3}{4}\left(5 \pm \sqrt{33}\right)$

Table 7.3: All solutions for $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, with $w_\pm = \frac{1}{4}\sqrt{(19 \pm 3\sqrt{33})}$.

7.4 Example: Sp$(n+1)/$Sp$(n) \equiv S^{4n+3}$

We can generalize our results for $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$ to coset spaces of the form $S^{4n+3} = \mathrm{Sp}(n{+}1)/\mathrm{Sp}(n)$. As we discussed in appendix A, $\mathrm{Sp}(n{+}1)/\mathrm{Sp}(n)$ is a space of dimension $4n+3$, where the "+3" dimensions correspond to a $\mathfrak{sp}(1)$ subalgebra. If we now identify this subalgebra with span$\{I_{a''}\}$, the structure constants for this case satisfy

$$f_{ia''b'} = f_{a''b''i} = f_{a''b''c'} = 0, \tag{7.68}$$

similarly to $\mathrm{SU}(n+1)/\mathrm{SU}(n)$. The situation here is a bit more complicated, since we no longer have $f_{a''b''c''} = 0$. This means that α_1'' is also non zero, in addition to α_2', α_4' and α_3''. Since a' takes $4n$ and a'' takes three different values the relation between α_2' and α_3'' for this case reads

$$4n\ \alpha_2' \ = \ 3\ \alpha_3''. \tag{7.69}$$

We can still calculate α_4' from α_2' using (6.46) and (6.46), but we no longer have $\alpha_3'' = 1$; instead, we get $\alpha_1'' + \alpha_3'' = 1$. Luckily, the value of α_1'' only depends on the normalization μ of the $I_{a''}$:

$$\alpha_1''\delta_{a''b''} \ = \ f_{a''c''d''}f_{b''c''d''} \ = \ 4\eta^2\varepsilon_{a''c''d''}\varepsilon_{b''c''d''} \ = \ 8\mu^2\delta_{a''b''} \tag{7.70}$$

where we used the canonical commutation relations for $\mathfrak{sp}(1) \equiv \mathfrak{su}(2)$. We have shown in the appendix that α_3'' satisfies a similar condition (equation (A.12)):

$$\alpha_3'' \ = \ 4n\ \mu^2 \tag{7.71}$$

which allows us to eliminate the normalization consant and determine all α's:

$$\alpha_2' \ = \ \frac{3}{4(n+2)}, \qquad \alpha_4' \ = \ \frac{2n+1}{4(n+2)}, \qquad \alpha_1' \ = \ \frac{2}{n+2}, \qquad \alpha_3'' \ = \ \frac{n}{n+2} \tag{7.72}$$

If we put these values into our differential equations, we get

$$\ddot{\phi} \ = \ \frac{1}{4(2+n)}\Big((2(1-n)+3\varkappa)\phi - 3(\varkappa+3)\phi\psi + 2(2+n)\phi^3 + 3\phi\psi^2\Big) \tag{7.73}$$

$$\ddot{\psi} \ = \ \frac{1}{8(2+n)}\Big(4(n+2)(\varkappa+1)\psi - 8(\varkappa+3)\psi^2 - 3(vk+3)\phi^2 + 16\psi^3 + 6\psi\phi^2\Big). \tag{7.74}$$

We will set $(\psi(\tau),\phi(\tau)) = \left(\Psi\left(\frac{1}{\sqrt{4(n+2)}}\tau\right), \sqrt{\frac{3}{4n}}\Phi\left(\frac{1}{\sqrt{4(n+2)}}\tau\right)\right)$, such that

$$\ddot{\Phi} \ = \ (2(1-n)+3\varkappa)\Phi - 3(\varkappa+3)\Phi\Psi + \frac{3(n+2)}{2n}\Phi^3 + 3\Phi\Psi^2 \tag{7.75}$$

$$\ddot{\Psi} \ = \ 2(n+2)(\varkappa+1)\Psi - 4(\varkappa+3)\Psi^2 - \frac{3(\varkappa+3)}{2}\Phi^2 + 8\Psi^3 + 3\Psi\Phi^2, \tag{7.76}$$

with the potential

$$\begin{aligned} V \;=\; & -(n+2)(\varkappa+1)\Psi^2 + \frac{4(\varkappa+3)}{3}\Psi^3 - 2\Psi^4 - \frac{3}{2}\Psi^2\Phi^2 \\ & + (n-1-\frac{3\varkappa}{2})\Phi^2 - \frac{3(n+2)}{8n}\Phi^4 + \frac{3(\varkappa+3)}{2}\Psi\Phi^2. \end{aligned} \tag{7.77}$$

7.4.1 Some solutions

Again, this more general situation can be analyzed analogouesly to the special case $n = 1$. For example, we get

$$\ddot{\Phi} = \frac{3(2+n)}{2n}\left(\Phi^3 - \frac{2n(2n+7)}{3(n+2)}\Phi\right), \qquad \text{for} \quad \varkappa = -3, \Psi = 0 \tag{7.78}$$

$$\ddot{\Psi} = 8\left(\Psi^3 - \frac{n+2}{2}\Psi\right), \qquad \text{for} \quad \varkappa = -3, \Phi = 0 \tag{7.79}$$

instead of (7.49) and (7.50), which is solved by

$$\Phi(\tau) = \sqrt{\frac{2n(2n+7)}{3(2+n)}}\tanh\left(\sqrt{\tfrac{7+2n}{2}}\tau\right), \qquad \text{for} \quad \varkappa = -3, \Psi = 0 \tag{7.80}$$

$$\Psi(\tau) = \sqrt{\tfrac{n+2}{2}}\tanh\left(\sqrt{2(n+2)}\tau\right), \qquad \text{for} \quad \varkappa = -3, \Phi = 0 \tag{7.81}$$

and

$$\ddot{\Phi} = \frac{3(n+2)}{2n}\left(\Phi^3 - \frac{4n}{3}\Phi\right), \quad \text{for} \quad \varkappa = -1, \Psi = 1 \tag{7.82}$$

with the solution

$$\Phi(\tau) = \sqrt{\frac{4n}{3}}\tanh\left(\sqrt{n+2}\tau\right), \quad \text{for} \quad \varkappa = -1, \Psi = 1. \tag{7.83}$$

The radial kink solutions we found before now arise at

$$\varkappa_\pm = \frac{3}{4}(3n + 2 \pm \sqrt{9n^2 + 20n + 4}) \tag{7.84}$$

and the equation for $\Phi = 0, \varkappa = \varkappa_\pm$ is

$$\ddot{\Psi} = 2(n+2)(\varkappa_\pm + 1)\Psi - 4(\varkappa_\pm + 3)\Psi^2 + 8\Psi^3. \tag{7.85}$$

This is solved by

$$\Psi_\pm(\tau) = -\frac{3(\varkappa_\pm+1)(n+2)}{4(\varkappa_\pm+3)}\left(\tanh\left(\sqrt{\frac{(\varkappa_\pm+1)(n+2)}{2}}\tau\right) - 1\right) \tag{7.86}$$

The solutions for the original equations (without rescaling), as well as for the analogoues cases (bounce- and periodic solutions) can be found in table 7.4.

Kink Solutions		
$\phi(\tau) = \tanh\left(\frac{1}{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{2n+7}{2n+4}}\tanh\left(\frac{1}{2}\sqrt{\frac{2n+7}{2n+4}}\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{\frac{n+2}{2}}\tanh\left(\sqrt{\frac{1}{2}}\tau\right)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{3(\varkappa+1)(n+2)}{4(\varkappa+3)}\left(\tanh\left(\frac{1}{2}\sqrt{\frac{(\varkappa+1)}{2}}\tau\right) - 1\right)$	$\phi = 0$	$\varkappa = \varkappa_{\pm}$
Bounce Solutions		
$\phi(\tau) = \sqrt{2}\,\mathrm{sech}\left(\frac{1}{2}\sqrt{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{2n+7}{n+2}}\,\mathrm{sech}\left(\frac{1}{2}\sqrt{\frac{2n+7}{n+2}}\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{n+2}\,\mathrm{sech}\,(\tau)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{3(\varkappa+1)(n+2)}{4(\varkappa+3)}\sqrt{2}\left(\mathrm{sech}\left(\frac{1}{2}\sqrt{\varkappa+1}\tau\right) - 1\right)$	$\phi = 0$	$\varkappa = \varkappa_{\pm}$
Periodic Kink Solutions		
$\phi(\tau) = N(k)k\ \mathrm{sn}_k\left(\frac{1}{2}N(k)\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{2n+7}{2n+4}}N(k)k\ \mathrm{sn}_k\left(\frac{1}{2}\sqrt{\frac{2n+7}{2n+4}}N(k)\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{\frac{n+2}{2}}N(k)k\ \mathrm{sn}_k\left(\sqrt{\frac{1}{2}}N(k)\tau\right)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{3(\varkappa+1)(n+2)}{4(\varkappa+3)}\left(N(k)k\ \mathrm{sn}_k\left(\frac{1}{2}\sqrt{\frac{(\varkappa+1)}{2}}N(k)\tau\right) - 1\right)$	$\phi = 0$	$\varkappa = \varkappa_{\pm}$
Periodic Bounce Solutions		
$\phi(\tau) = \sqrt{2}M(k)k\ \mathrm{cn}_k\left(\frac{1}{2}\sqrt{2}\tau\right)$	$\psi = 1$	$\varkappa = -1$
$\phi(\tau) = \sqrt{\frac{2n+7}{n+2}}M(k)k\ \mathrm{cn}_k\left(\frac{1}{2}\sqrt{\frac{2n+7}{n+2}}\tau\right)$	$\psi = 0$	$\varkappa = -3$
$\psi(\tau) = \sqrt{n+2}M(k)k\ \mathrm{cn}_k\,(\tau)$	$\phi = 0$	$\varkappa = -3$
$\psi(\tau) = -\frac{3(\varkappa+1)(n+2)}{4(\varkappa+3)}\sqrt{2}\left(M(k)k\ \mathrm{cn}_k\left(\frac{1}{2}\sqrt{\varkappa+1}\tau\right) - 1\right)$	$\phi = 0$	$\varkappa = \varkappa_{\pm}$

Table 7.4: All solutions for $S^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$.

8 Conclusion

This thesis goal was to find solutions to the torsion-full Yang-Mills equations. This was realized on spaces $\mathbb{R} \times G/H$, where the Cartan-Killing metric allows one to always define a totally antisymmetric torsion proportional to the structure constants. Choosing a G-invariant ansatz for the gauge connection on the principal bundle $((\mathbb{R} \times G/H) \times G, \pi, \mathbb{R} \times G/H)$ allowed the simplification of the Yang-Mills equations to differential equations for matrices X_a. Finally, assuming that the tangent space of G/H could be decomposed into two distinct sets lead to equations which are similar to those describing the movement of a single particle moving in a two-dimensional potential. This allowed the construction of solutions on the odd-dimensional spheres $S^{n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ as well as $S^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$.

The kind of solutions found are quite common in classical, non-abelian field theories; since they have a localized energy density and are topological stable (they cannot be continously transformed into the vacuum state of the theory), they are commonly interpreted as particles. And although these are classical solutions, their particle like properties are usually not lost under quantization.

The original motivation for studying the torsion-full Yang-Mills equations was that they appear as part of the field eqations of heterotic supergravity. It would be interesting to explore how our solutions could be integrated into this framework, similar to the procedure in [25].

Another possible way to generalize the results would be to consider coset spaces where the the tangent space of G/H splits into more than two parts, leading to more differential equations.

A The group Sp(n) and its Lie algebra

A.1 Quaternions

The compact symplectic group, or Sp(n), is defined as the group of unitarian quaternionic matrices. The quaternions are $\mathbb{R}^4$ with the usual addition and scalar multiplication but equipped with an additional multiplication such that the four base elements $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ satisfy the identities

$$\mathrm{i}^2 \;=\; \mathrm{j}^2 \;=\; \mathrm{k}^2 \;=\; \mathrm{ijk} \;=\; -1 \tag{A.1}$$

which implies the useful properties

$$\mathrm{ij} \;=\; \mathrm{k}, \qquad \mathrm{jk} \;=\; \mathrm{i} \qquad \mathrm{ki} \;=\; \mathrm{j}. \tag{A.2}$$

The quaternionic multiplication is antisymmetric for $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ (e.g., $\mathrm{ij} = -\mathrm{ji}$).

Similar to the complex numbers, one can define the conjugate of a quaternion $q = a1 + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ as

$$g^* \;=\; a1 - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}. \tag{A.3}$$

This allows us to define the adjoint of a quaternionic matrix analogously to the adjoint of a complex matrix by transposing the matrix and taking the conjugate of each entry.

Although quaternions allow a rather compact notation, it is often more intuitive to work with a matrix representation of the quaternionic algebra, such that the quaternionic multiplication corresponds to matrix multiplication and the conjugate corresponds to the usual adjoint of the matrix. Such a representation is given by

$$\{\mathbb{1}, \mathrm{i}_{\mathbb{C}}\sigma_1, \mathrm{i}_{\mathbb{C}}\sigma_2, \mathrm{i}_{\mathbb{C}}\sigma_3\} \tag{A.4}$$

where σ_i are the usual pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i}_{\mathbb{C}} \\ \mathrm{i}_{\mathbb{C}} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5}$$

and $\mathrm{i}_{\mathbb{C}}$ is the complex unit in $\mathbb{C}$ (we will drop the subscript when it is obvious from the context which i is used).

A.2 The Lie algebra $\mathfrak{sp}(\mathfrak{n})$

Due to the similarity of the complex numbers and the quaternions, the Lie algebra of Sp(n) can be derived in a similar manner to $\mathfrak{su}(n)$; it is simply given by

$$\mathfrak{sp}(n) = \left\{ A \in \mathrm{Mat}(n \times n, \mathbb{H}) \mid A + A^{\dagger} = 0 \right\} \tag{A.6}$$

(i.e., the anti hermitian quaternionic matrices). We can now explicitly construct a basis for $\mathfrak{sp}(n)$ by following the same recipe that was used to construct a basis for $\mathfrak{su}(n)$, with the only difference that when ever we write down a matrix containing an $\mathrm{i}_{\mathbb{C}}$ we now have to write down three matrices, one for i, j and k. For example, a basis for $\mathfrak{sp}(1)$ is simply given by

$$\{\mathrm{i}, \mathrm{j}, \mathrm{k}\} \equiv \left\{ \begin{pmatrix} 0 & \mathrm{i}_{\mathbb{C}} \\ \mathrm{i}_{\mathbb{C}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i}_{\mathbb{C}} & 0 \\ 0 & -\mathrm{i}_{\mathbb{C}} \end{pmatrix} \right\}, \tag{A.7}$$

which immediatly shows that $\mathfrak{su}(2) \equiv \mathfrak{sp}(1)$. This allow us to directly write down a basis for $\mathfrak{sp}(2)$, by taking the complex matrices in (A.7) and replacing $\mathrm{i}_{\mathbb{C}}$ with $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$:

$$\mathfrak{sp}(2) = \mathrm{span}\Big\{ \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{k} \\ \mathrm{k} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.8}$$

$$\begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{j} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{k} & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.9}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix} \Big\} \tag{A.10}$$

Notice that we got six diagonal matrices instead of just three because the anti hermicity condition does allow for mixed entries on the diagonal. We observe that both (A.9) and (A.10) are just the generators of $\mathfrak{sp}(1)$ embedded into one higher dimension (i.e., they both span a $\mathfrak{sp}(1)$ sub algebra).

The basis for $\mathfrak{sp}(n+1)$ can now be constructed similary to above: Embed the generators of $\mathfrak{sp}(n)$ into one higher dimension and add three matrices with just $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ in the last diagonal and four matrices similar to (A.8) for every new row, meaning the generators of $\mathfrak{sp}(n+1)$ are given by matrices of the form

$$I_i = \left(\begin{array}{c|c} \mathfrak{sp}(n) & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline 0\ 0 \cdots 0 & 0 \end{array} \right), I_{a''} = \left(\begin{array}{c|c} 0 & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline 0\ 0 \cdots 0 & \mathfrak{sp}(1) \end{array} \right), I_{a'} = \left(\begin{array}{c|c} 0 & \begin{array}{c} 0 \\ \alpha \\ 0 \\ \vdots \end{array} \\ \hline 0\ \beta\ 0 \cdots & 0 \end{array} \right)$$

where α, β are choosen as in (A.8) such that the matrix is anti hermitian. This construction implies that the Lie algebra of $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ will always decompose into a $\mathfrak{sp}(1)$ sub algebra and a $4n$-dimensional space spanned by the new non-diagonal matrices, leading to a total dimension of $(4n+3)$. Knowing that $\mathrm{Sp}(1)$ has dimension three, this implies that $\mathrm{Sp}(n)$ has total dimension $n(2n+1)$.

It is now a simple application of linear algebra to show that the generators I_i, $I_{a'}$ and $I_{a''}$ satisfy the commutation relations

$$f_{ia''b'} = f_{a''b''i} = f_{a''b''c'} = 0, \tag{A.11}$$

expressed using the structure constants. If we further assume that the generators $I_{a''}$ are normalized with some constant μ, one can use either (A.2) or the canonical commutation relations for $\mathfrak{su}(2)$ to show that

$$\sum_{c',d'} f_{a''c'd'} f_{a''c'd'} = 4n\mu^2. \tag{A.12}$$

$\mathcal{B}$

G-invariance condition

In order to describe G-invariant connections, we need a theorem from [23, section II-11]. For this, let K be a group of automorphisms of the principal G-bundle $(\mathcal{P}, \pi, \mathcal{M})$. Let u_0 be an arbitrary, but fixed point of $\mathcal{P}$, and let J be the closed subgroup of K that fixes $x_0 = \pi(u_0)$. This means that for every $j \in J$ there is an $a \in G$ such that $ju_0 = u_0 a$, since ju_0 is in the same fibre as u_0. Now define the homomorphism $\lambda : J \to G$ by $\lambda(j) = a$. Taking the derivative gives an induced Lie algebra homomorphism, which we will also denote by $\lambda : \mathfrak{j} \to \mathfrak{g}$. Make the further assumption that the Lie algebra $\mathfrak{k}$ of K can be decomposed into $\mathfrak{k} = \mathfrak{j} \oplus \mathfrak{m}$, such that $\mathfrak{Ad}(J)(\mathfrak{m}) = \mathfrak{m}$. Then the following holds true:

Theorem B.0.1. *There is a one-to-one correspondence between the sets of K-invariant connections on $\mathcal{P}$ and the set of linear mappings $\Lambda_\mathfrak{m} : \mathfrak{m} \to \mathfrak{g}$ such that*

$$\Lambda_\mathfrak{m}(\mathfrak{Ad}(j)(X)) \;=\; \mathfrak{Ad}(\lambda(j))(\Lambda_\mathfrak{m}(X)), \forall X \in \mathfrak{m}, j \in J. \tag{B.1}$$

The correspondence is given by

$$\Lambda(X) \;=\; \omega_{u_0}(\tilde{X}), \tag{B.2}$$

where ω is the connection one form, $\tilde{X}$ is the vector field on $\mathcal{P}$ induced by $X \in \mathfrak{k}$ and Λ is given by

$$\Lambda(X) = \begin{cases} \lambda(X) & \text{if } X \in \mathfrak{j} \\ \Lambda_\mathfrak{m}(X) & \text{if } X \in \mathfrak{m} \end{cases} \tag{B.3}$$

In our situation we consider the trivial principal fibre bundle $((\mathbb{R} \times G/H) \times G, \pi, \mathbb{R} \times G/H)$ with structure group G. Since H is a closed subgroup of G the Lie algebra of $\mathfrak{g}$ decomposes into

$$\mathfrak{g} \;=\; \mathfrak{m} \oplus \mathfrak{h} \tag{B.4}$$

If we now identify $J \equiv H$ and $K \equiv G$ we have the situation described above. Since H is just a subgroup of the gauge group G, the homomorphism λ is trivial. Using the notation used in chapter 6, this means that the connection

$$\mathcal{A} \;=\; X_i e^i + X_a e^a \tag{B.5}$$

is given by

$$X_i := \Lambda(I_i) \;=\; \lambda(I_i) \;=\; I_i \tag{B.6}$$

and

$$X_a := \Lambda(I_a) \;=\; X_a^B I_B \;=\; X_a^i I_i + X_a^b I_b \;=\; X_a^b I_b, \tag{B.7}$$

if we assume that $X_a^i = 0$. We still need to evaluate (B.1). For this, consider $X = I_a$ and $j = \exp(tI_i)$:

$$(B.1) \implies \Lambda\left(\mathfrak{Ad}(\exp(tI_i)(I_a)\right) \;=\; \mathfrak{Ad}(\exp(tI_i))(\Lambda(I_a)) \tag{B.8}$$

The adjoint representation was defined by $\mathfrak{Ad}(g) = (\mathrm{Ad}_g)_{*\mathbb{1}}$. If we write $\Lambda(I_a) = X_a = \frac{\partial}{\partial_s} \exp sX_a\Big|_{s=0}$, the right hand side is given by

$$\mathfrak{Ad}(\exp(tI_i))(\Lambda(I_a)) \;=\; \frac{\partial}{\partial s}\left(\exp(tI_i)\exp(sX_a)\exp(-tI_i)\right)\Big|_{s\,=\,0}. \tag{B.9}$$

If we differentiate this with respect to t we get the commutator $[I_i, X_a]$. For the left hand side, we get

$$\Lambda(\mathfrak{Ad}(\exp(tI_i))(I_a)) \;=\; \Lambda\left(\frac{\partial}{\partial s}\left(\exp(tI_i)\exp(sI_a)\exp(-tI_i)\right)\Big|_{s\,=\,0}\right). \tag{B.10}$$

Since Λ is linear, differentiating this yields

$$\begin{aligned}
\frac{\partial}{\partial t}\Lambda(\mathfrak{Ad}(\exp(tI_i))(I_a))\Big|_{t\,=\,0} \;&=\; \Lambda\left(\frac{\partial}{\partial t}\mathfrak{Ad}(\exp(tI_i))(I_a))\Big|_{t\,=\,0}\right) &\text{(B.11)}\\
&=\; \Lambda\left([I_i, I_a]\right) &\text{(B.12)}\\
&=\; \Lambda\left(f_{ia}^b I_b\right) &\text{(B.13)}\\
&=\; f_{ia}^b \Lambda\left(I_b\right) &\text{(B.14)}\\
&=\; f_{ia}^b X_b. &\text{(B.15)}
\end{aligned}$$

This means that the G-invariance of $\mathcal{A}$ implies that the X_a have to satisfy

$$[I_i, X_a] \;=\; f_{ia}^b X_b, \tag{B.16}$$

which is the condition imposed in (6.21).

C

Yang-Mills equations on α-Sasakian manifolds

Suppose $\mathcal{M} = S^5 \equiv \mathrm{SU}(3)/\mathrm{SU}(2)$, equipped with generators $\tilde{I}_a, \tilde{I}_i$ as in chapter 6, such that the Cartan-Killing metric is given by $\tilde{g}_{ab} = \delta_{ab}$. Than $\mathcal{M}$ is an α-Sasakian manifold with $\alpha = -\frac{1}{2}$ (cf. [13,14]). It has been argued in the afformentioned paper that different values of α can be achieved by rescaling the generators $\tilde{I}_a$, and that such a rescaling propagates to a rescaling of the basis $\{E_a\}, \{e^a\}$. And although equations (6.40) are invariant under such a rescaling, this invariance is not manifest in the derived equations (6.54),(6.55) for ϕ and ψ, since we explicitly used that $\tilde{g}_{ab} = \delta_{ab}$.

Assume now that we split the indices as in section 7.1 (that is, $\{a'\} = \{2, 3, 4, 5\}$, $\{a''\} = \{1\}$), and let the metric satisfy

$$g_{a'B} = \frac{1}{\beta^2}\delta_{a'B}, \qquad g_{1B} = \frac{1}{\gamma^2}\delta_{1B}. \tag{C.1}$$

This corresponds to a rescaling

$$\tilde{I}_{a'} = \beta I_{a'}, \qquad \tilde{I}_1 = \gamma I_1 \tag{C.2}$$

of the original generators $\tilde{I}_a$, which implies an α-Sasakian structure with

$$\alpha = -\frac{\gamma}{2\beta^2}. \tag{C.3}$$

The structure constants for the new generators still satisfy

$$f^{c'}_{a'b'} = f^1_{ia'} = f^A_{11} = f^a_{ij} = 0, \tag{C.4}$$

which means that the metric[1] is given by

$$g_{a'b'} = \underbrace{2f^{1}_{a'c'}f^{c'}_{1b'}}_{:=\alpha'} + \underbrace{2f^{i}_{a'c'}f^{c'}_{ib'}}_{:=\beta^{-2}-\alpha'}, \qquad g_{11} = \underbrace{2f^{d'}_{1c'}f^{c'}_{d'1}}_{:=\alpha''}. \tag{C.5}$$

Using this metric, the structure constants f^{c}_{ab} are no longer totally antisymmetric; instead, we get

$$f^{c'}_{a1} = -\tfrac{\beta^2}{\gamma^2} f^{1}_{ac'} \tag{C.6}$$

when exchanging "mixed" indices.

We can now re-derive equations (6.54),(6.55) for this situation, starting from (6.40):

$$\begin{aligned}
\ddot{X}^{c'} &= \left(\tfrac{1}{2}(\varkappa+1)f^{c'}_{ab}f^{abe} + f^{ac'j}f^{e}_{aj}\right)X_e - \tfrac{1}{2}(\varkappa+1)f^{c'}_{ab}[X^a, X^b] \\
&\quad + f^{ac'b}[X_a, X_b] - [X_a, [X^a, X^{c'}]] \\
&= \left(\tfrac{1}{2}(\varkappa+1)(f^{c'}_{a'b'}f^{a'b'e'} + 2f^{c'}_{a'1}f^{a'1e'} + f^{c'}_{11}f^{11e'}) + f^{a'c'j}f^{e'}_{a'j} + f^{1c'j}f^{e'}_{1j}\right)X_{e'} \\
&\quad + \left(\tfrac{1}{2}(\varkappa+1)(f^{c'}_{a'b'}f^{a'b'1} + 2f^{c'}_{a'1}f^{a'11} + f^{c'}_{11}f^{111}) + f^{a'c'j}f^{1}_{a'j} + f^{1c'j}f^{1}_{1j}\right)X_1 \\
&\quad - \tfrac{1}{2}(\varkappa+1)\left(f^{c'}_{a'b'}[X^{a'}, X^{b'}] + 2f^{c'}_{a'1}[X^{a'}, X^1] + f^{c'}_{11}[X^1, X^1]\right) \\
&\quad + f^{a'c'b'}[X_{a'}, X_{b'}] + 2f^{a'c'1}[X_{a'}, X_1] + f^{1c'1}[X_1, X_1] \\
&\quad - [X_{a'}, [X^{a'}, X^{c'}]] - [X_1, [X^1, X^{c'}]].
\end{aligned} \tag{C.7}$$

A lot of the terms vanish. If we also pull down some of the indices to get the structure constants into their natural index position, we get

$$\begin{aligned}
\beta^2\ddot{X}_{c'} &= \left((\varkappa+1)\beta^2\gamma^2 f^{c'}_{a'1}f^{e'}_{a'1} + \beta^4 f^{j}_{a'c'}f^{e'}_{a'j}\right)X_{e'} \\
&\quad - (\varkappa+1)\beta^2\gamma^2 f^{c'}_{a'1}[X_{a'}, X_1] + 2\beta^4 f^{1}_{a'c'}[X_{a'}, X_1] \\
&\quad - \beta^4[X_{a'}, [X_{a'}, X_{c'}]] - \beta^2\gamma^2[X_1, [X_1, X_{c'}]] \\
&= \left((\varkappa+1)\beta^4 f^{1}_{c'a'}f^{a'}_{1e'} - \beta^4 f^{j}_{c'a'}f^{a'}_{je'}\right)X_{e'} \\
&\quad + (\varkappa+1)\beta^4 f^{1}_{a'c'}[X_{a'}, X_1] + 2\beta^4 f^{1}_{a'c'}[X_{a'}, X_1] \\
&\quad - \beta^4[X_{a'}, [X_{a'}, X_{c'}]] - \beta^2\gamma^2[X_1, [X_1, X_{c'}]] \\
&= \tfrac{1}{2}\beta^2\left((\varkappa+2)\beta^2\alpha' - 1\right)X_{c'} \\
&\quad + (\varkappa+3)\beta^4 f^{1}_{a'c'}[X_{a'}, X_1] \\
&\quad - \beta^4[X_{a'}, [X_{a'}, X_{c'}]] - \beta^2\gamma^2[X_1, [X_1, X_{c'}]]
\end{aligned}$$

If we make an ansatz for X_a analogous to the one in section 6.3, that is

$$X_{a'} = \Phi I_{a'}, \qquad X_1 = \Psi I_1, \tag{C.8}$$

[1] Here, α' and α'' do not refer to the α-Sasaki structure; they are defined analogous to section 6.3, where we have the freedom to pull the indices up with $\tilde{g}_{ab} = \delta_{ab}$.

our equations for this case read

$$\begin{aligned}
\ddot{\Phi} I_{a'} &= \tfrac{1}{2}\beta^2\left((\varkappa+2)\beta^2\alpha'-1\right)\Phi I_{a'} \\
&\quad + (\varkappa+3)\beta^2 f^1_{a'c'}[\Phi I_{a'}, \Psi I_1] \\
&\quad - \beta^2[\Phi I_{a'}, [\Phi I_{a'}, \Phi I_{c'}]] - \gamma^2[\Psi I_1, [\Psi I_1, \Phi I_{c'}]] \\
&= \tfrac{1}{2}\left((\varkappa+2)\beta^2\alpha'-1\right)\Phi I_{a'} \\
&\quad + (\varkappa+3)\beta^2 \underbrace{f^1_{a'c'} f^{e'}_{a'1}}_{=-\frac{1}{2}\alpha'\delta_{c'e'}} \Phi\Psi I_{e'} \\
&\quad - \beta^2 \underbrace{(f^i_{a'c'} f^{e'}_{a'i} + f^1_{a'c'} f^{e'}_{a'1})}_{=-\frac{1}{2\beta^2}\delta_{c'e'}} \Phi^3 I_{e'} - \gamma^2 \underbrace{f^{d'}_{1c'} f^{e'}_{1d'}}_{=-\frac{\beta^2}{2\gamma^2}\alpha'\delta_{c'e'}} \Psi^2\Phi I_{e'}
\end{aligned}$$

which is equivalent to

$$\ddot{\Phi} = \frac{1}{2}\left((\varkappa+2)\beta^2\alpha'-1\right)\Phi - \frac{\varkappa+3}{2}\beta^2\alpha'\Phi\Psi + \frac{1}{2}\Phi^3 + \frac{1}{2}\beta^2\alpha'\Psi^2\Phi. \tag{C.9}$$

The α' appearing in (C.9) is the one for the new basis; it can be calculated from the one in the old basis (here, $\tilde{f}^c_{ab}$ are the structure constants with respect to the original basis, no sum over c'):

$$\tilde{\alpha}' \overset{(7.4)}{=} \frac{1}{2} = 2\tilde{f}^1_{c'a'}\tilde{f}^{a'}_{1c'} = 2\beta^2 f^1_{c'a'} f^{a'}_{1c'} = \beta^2\alpha', \tag{C.10}$$

which means that we indeed get our original equation (7.5),

$$\ddot{\Phi} = \frac{\varkappa}{4}\Phi - \frac{\varkappa+3}{4}\Phi\Psi + \frac{1}{2}\Phi^3 + \frac{1}{4}\Psi^2\Phi. \tag{C.11}$$

The equation for Ψ can be derived analogously.

D

Kink Equations

D.1 Transverse and radial kink

All the analytical solutions we find arise from some variation of the kink-equation

$$\ddot{\Phi} = 2(\Phi^3 - \Phi) \tag{D.1}$$

which has the non-periodic solution

$$\Phi(\tau) = \tanh \tau. \tag{D.2}$$

This result can be generalized by considering a rescaled function

$$\Phi(\tau) = \mu \tanh(\lambda\tau), \tag{D.3}$$

which then solves the more general equation

$$\ddot{\Phi} = 2\tfrac{\lambda^2}{\mu^2}(\Phi^3 - \mu^2\Phi). \tag{D.4}$$

Solutions of this form will always mediate between two values $\pm\Phi_{max}$; they are called transverse kinks. We will also encounter situations where we expect solutions that run from 0 to Φ_{max} (i.e., a radial kink). These will be of the form

$$\Phi(\tau) = \mu(\tanh(\lambda\tau) - 1), \tag{D.5}$$

solving the differential equation

$$\ddot{\Phi} = 2\tfrac{\lambda^2}{\mu^2}(\Phi^3 + 3\mu\Phi^2 + 2\mu^2\Phi). \tag{D.6}$$

Notice that this equation has only two free parameters, which means that not every third order polynomial can be brought into this form.

D.2 Periodic kink solutions

The hyperbolic tangens is actually the non-periodic limit $k \to 1$ of the more general solution to equation (D.1)

$$\Phi(\tau) \;=\; N(k)k \; \mathrm{sn}_k\left(N(k)\tau\right) \tag{D.7}$$

where we set $N(k) := \sqrt{\frac{2}{1+k^2}}$.

It has a periodicity of

$$L = 2\sqrt{2(1+k^2)}K_k \tag{D.8}$$

where K_k is the elliptic integral

$$K_k \;=\; \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2(\varphi)}}. \tag{D.9}$$

This means that we can interpret our equations as equations on $S^1 \times G/H$ instead of $\mathbb{R} \times G/H$, where we then find periodic solutions. These can be understood as chains of alternating kinks and anti-kinks, equally spaced around the circle. See [29, chapter 11.2] for further information on these solutions and their physical interpretation. Similarly to above, we will have to rescale our function to fit our equations, meaning that

$$\ddot{\Phi} \;=\; 2\tfrac{\lambda^2}{\mu^2}(\Phi^3 - \mu^2\Phi). \tag{D.10}$$

has the general (periodic) solution

$$\Phi(\tau) \;=\; \mu N(k)k \; \mathrm{sn}_k\left(\lambda N(k)\tau\right) \tag{D.11}$$

D.3 Bounce solutions

We can also consider the case where our metric has Lorentzian signature. This corresponds to a transformation

$$\tau \to \mathrm{i}\tau \tag{D.12}$$

which flips the sign of our differential equations. These new equations will then be variations of

$$\ddot{\Phi} \;=\; -2(\Phi^3 + \Phi) \tag{D.13}$$

which has the non-periodic solution

$$\Phi(\tau) \;=\; \sqrt{2}\;\mathrm{sech}(\sqrt{2}\tau) \tag{D.14}$$

This means that we simply need to replace $\tanh(\tau) \to \sqrt{2}\,\mathrm{sech}(\sqrt{2}\tau)$ in our original solutions to get solutions for this case. This applies to the rescaled function as well, meaning that

$$\Phi(\tau) \;=\; \sqrt{2}\mu \;\; \mathrm{sech}(\lambda\sqrt{2}\tau) \tag{D.15}$$

solves

$$\ddot{\Phi} \;=\; -2\tfrac{\lambda^2}{\mu^2}(\Phi^3 - \mu^2\Phi). \tag{D.16}$$

The difference between Kink and Bounce solutions is that the latter will no longer mediate between two critical points; Since the hyperbolic secant takes the same value for $\tau \to \pm\infty$, they will instead start at the origin (or some other value, see below), "bouncl across a critical point, and than return to their inital value. To describe a bounce from a different value, one can look at

$$\Phi(\tau) \;=\; \mu(\sqrt{2}\,\mathrm{sech}(\lambda\sqrt{2}\tau) - 1), \tag{D.17}$$

solving the differential equation

$$\ddot{\Phi} \;=\; -2\tfrac{\lambda^2}{\mu^2}(\Phi^3 + 3\mu\Phi^2 + 2\mu^2\Phi). \tag{D.18}$$

D.4 Periodic bounce solutions

Just as the hyperbolic tangens is the non-periodic limit of the Jacobi elliptic function sn, the hyperbolic secant is the non-periodic limit $k \to 1$ of the general solution to equation (D.13):

$$\Phi(\tau) \;=\; \sqrt{2}M(k)k\ \mathrm{cn}_k\left(\sqrt{2}M(k)\tau\right) \tag{D.19}$$

now with $M(k) := \sqrt{\frac{1}{2k^2-1}}$.

It has the periodicity

$$L = 2\sqrt{4k^2 - 2}K_k. \tag{D.20}$$

This means that (D.15) has the general solution

$$\Phi(\tau) \;=\; \mu\sqrt{2}M(k)k\ \mathrm{cn}_k\left(\lambda\sqrt{2}M(k)\tau\right). \tag{D.21}$$

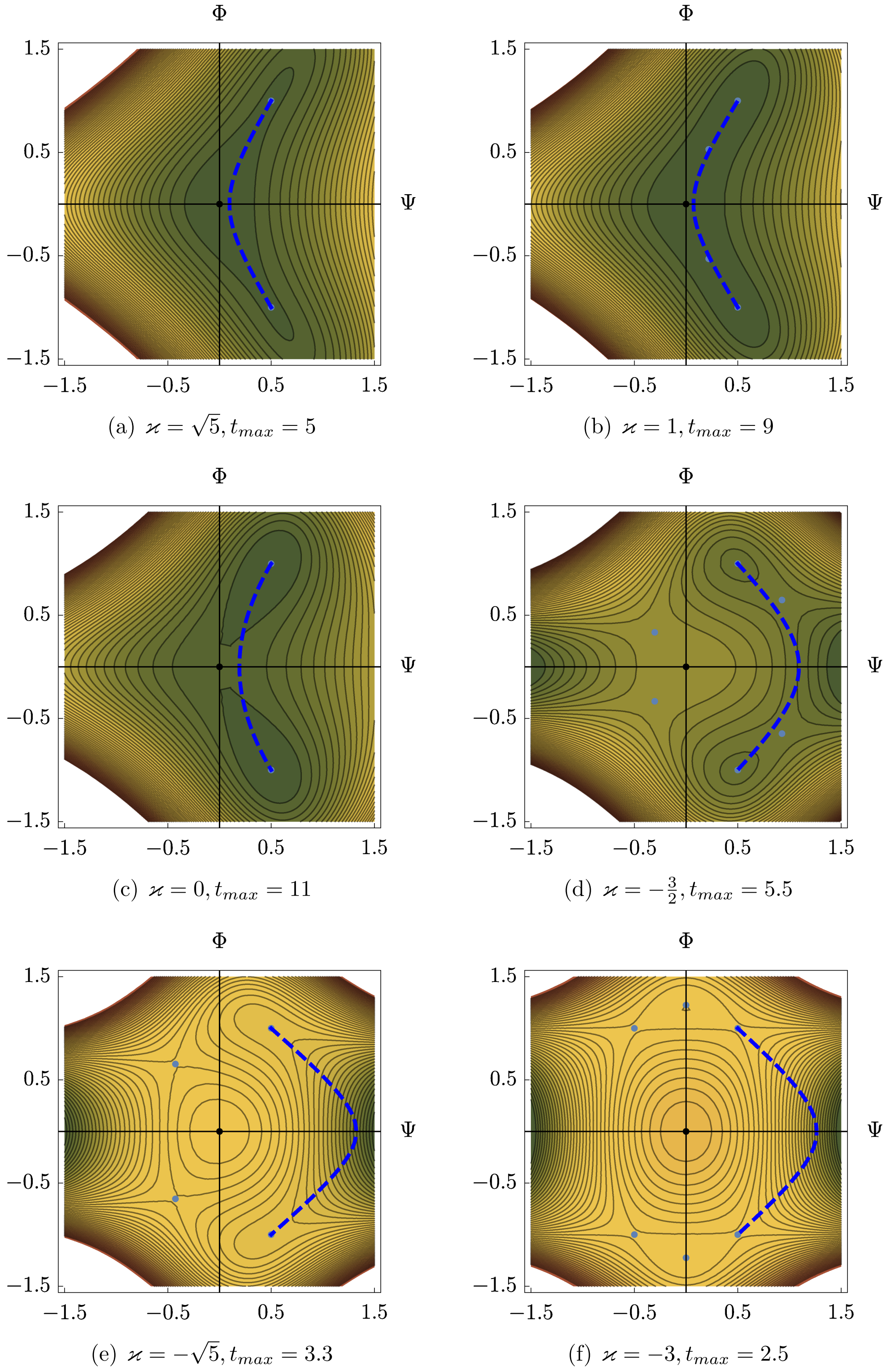
$\mathcal{E}$ Numerical Solutions

All numerical solutions were created with Mathematica 10.0 using code similar to this:

```
(*RHS of the differential equations for phi, psi*)
Phidotdot[Psi_, Phi_, k_, ap_, apone_, aptwo_, apthree_, apfour_,
   apfive_] := (1/2 (k + 2) ap - 1/2) Phi -
   1/2 (k + 3) (Phi^2 apone + 2 Phi*Psi*aptwo + Psi^2 apthree) +
   Phi^3 (apone + aptwo + apfour) +
   Phi*Psi^2 (aptwo + apthree + apfive);
Psidotdot[Psi_, Phi_, k_, ap_, apone_, aptwo_, apthree_, apfour_,
   apfive_] := (1/2 (k + 2) ap - 1/2) Psi -
   1/2 (k + 3) (Psi^2 apone + 2 Phi*Psi*aptwo + Phi^2 apthree) +
   Psi^3 (apone + aptwo + apfour) +
   Psi*Phi^2 (aptwo + apthree + apfive);

(* input values for all alpha',alpha'' *)
aprimes = {3/(2 (n + 2)), 0, 3/(4 (n + 2)), 0, (2 n + 1)/(4 (n + 2)),
   0};
adprimes = {1, 2/(n + 2), 0, (n)/(n + 2), 0, 0};

(* Rescalings tau -> resc[1]tau,
 psi -> resc[2]psi, phi -> resc[3]phi *)

resc = {Sqrt[1/(4 (n + 2))], 1, Sqrt[3/(4 n)]};

(* Rescaled equations for these values *)
PhiSpn[Psi_, Phi_, k_, n_] :=
  1/resc[[1]]^2*
```

```
   Phidotdot[resc[[2]] Psi, resc[[3]] Phi, k, aprimes[[1]],
     aprimes[[2]], aprimes[[3]], aprimes[[4]], aprimes[[5]],
     aprimes[[6]]]/resc[[3]];
PsiSpn[Psi_, Phi_, k_, n_] :=
  1/resc[[1]]^2*
   Psidotdot[resc[[2]] Psi, resc[[3]] Phi, k, adprimes[[1]],
     adprimes[[2]], adprimes[[3]], adprimes[[4]], adprimes[[5]],
     adprimes[[6]]]/resc[[2]];

(* Specify starting and end point {psi,phi} for numerical solution,
as well as \[Kappa],n and maximal t parameter*)
{psi0,phi0} = {1, -Sqrt[4/3]};
{psi1, phi1} = {1, Sqrt[4/3]};
k = 3/5;
n = 1;
tmax = 2.3;

(* solve equations *)
sol =
 NDSolve[{Phi''[t] == PhiSpn[Psi[t], Phi[t], k, n],
   Psi''[t] == PsiSpn[Psi[t], Phi[t], k, n], Phi[0] == phi0,
   Psi[0] == psi0, Phi[tmax] == phi1, Psi[tmax] == psi1}, {Phi,
   Psi}, {t, 0, tmax},
  Method -> {"Shooting",
    "StartingInitialConditions" -> {Phi'[0] == 0, Psi'[0] == 0}}]
```

The axis labels in the illustrations where created utilizing the *MaTex*-package, available at `https://github.com/szhorvat/MaTeX`.



(a) $\varkappa = \sqrt{5}, t_{max} = 5$

(b) $\varkappa = 1, t_{max} = 9$

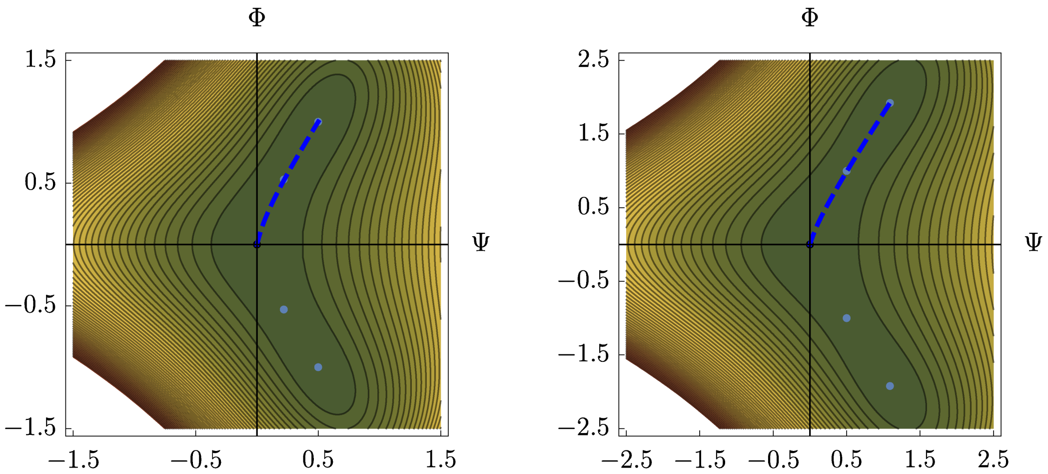
(c) $\varkappa = 0, t_{max} = 11$

(d) $\varkappa = -\frac{3}{2}, t_{max} = 5.5$

(e) $\varkappa = -\sqrt{5}, t_{max} = 3.3$

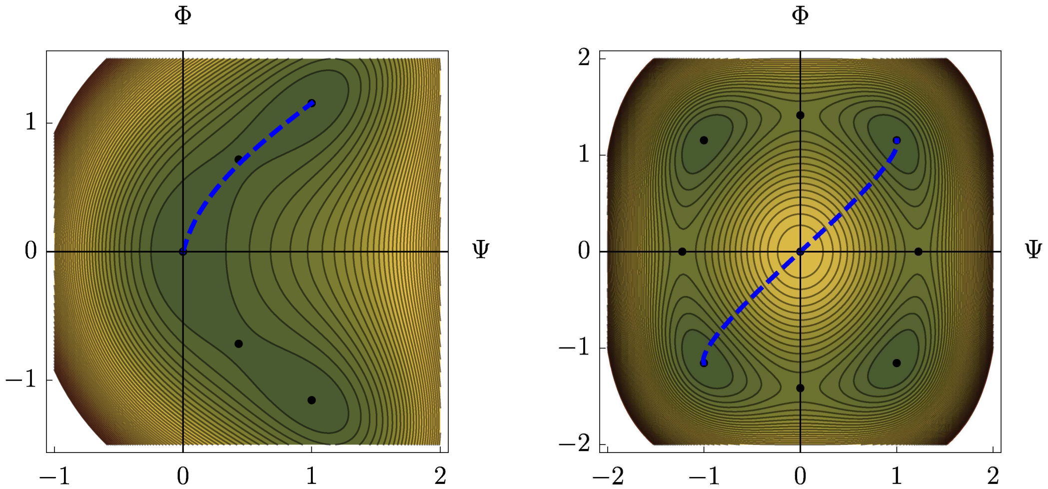
(f) $\varkappa = -3, t_{max} = 2.5$

Figure E.1: Numerical Solutions for $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ between the two fixed critical points at $(\Psi, \Phi) = (\frac{1}{2}, \pm 1)$ for various values of $\varkappa$.



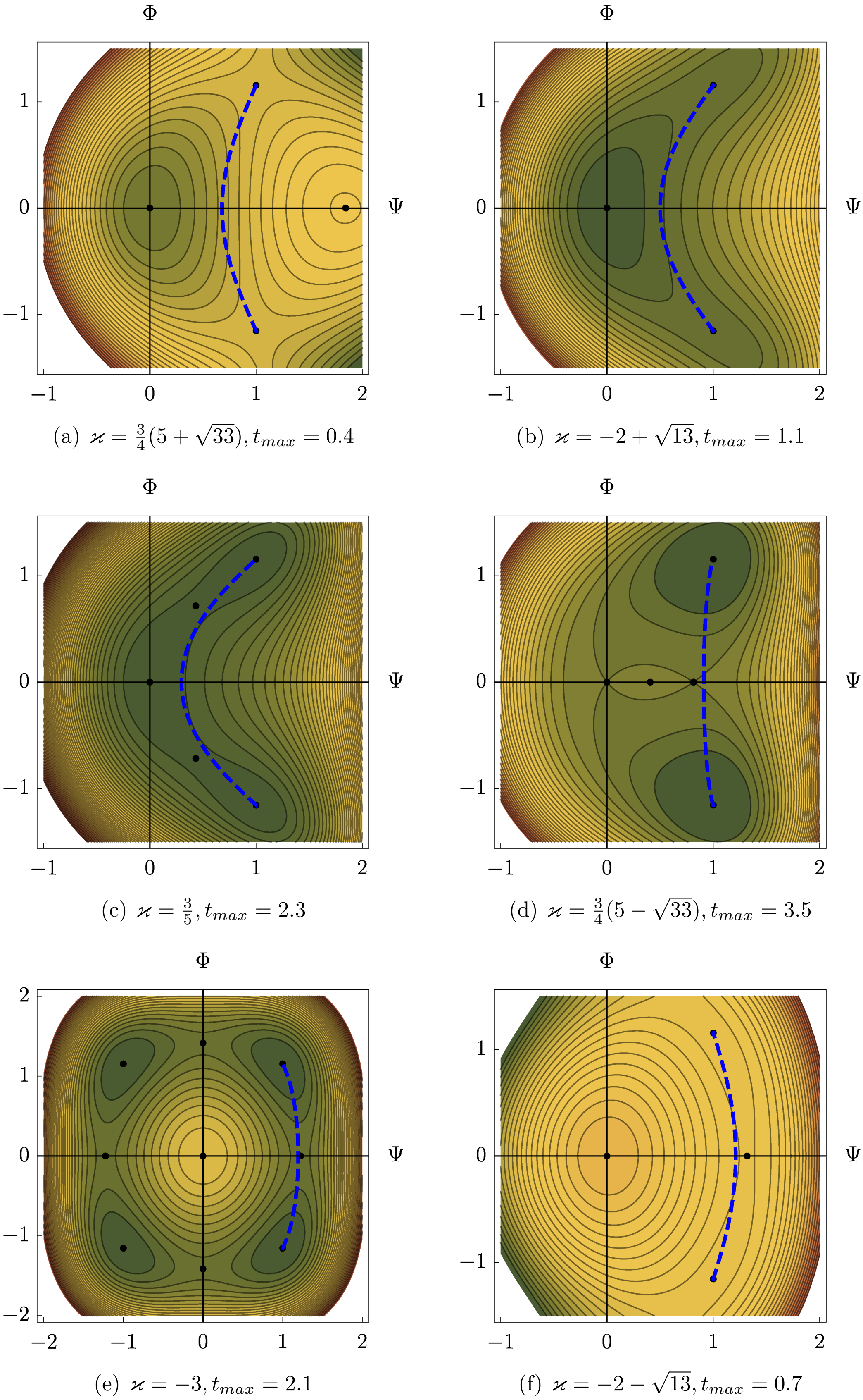
(a) $\varkappa = 1, t_{max} = 7.2$ (b) $\varkappa \approx 4.75, t_{max} = 4.6$

Figure E.2: Numerical Solutions for $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ starting at the origin.



(a) $\varkappa = \frac{3}{5}, t_{max} = 2.3$ (b) $\varkappa = -3, t_{max} = 3.8$

Figure E.3: Numerical Solutions for $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$.



(a) $\varkappa = \frac{3}{4}(5+\sqrt{33}), t_{max} = 0.4$

(b) $\varkappa = -2+\sqrt{13}, t_{max} = 1.1$

(c) $\varkappa = \frac{3}{5}, t_{max} = 2.3$

(d) $\varkappa = \frac{3}{4}(5-\sqrt{33}), t_{max} = 3.5$

(e) $\varkappa = -3, t_{max} = 2.1$

(f) $\varkappa = -2-\sqrt{13}, t_{max} = 0.7$

Figure E.4: Numerical Solutions for $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$ between the two fixed critical points at $(\Psi, \Phi) = (1, \pm\sqrt{\frac{4}{3}})$ for various values of $\varkappa$.

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